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# A nonlinear semigroup approach to Hamilton-Jacobi equations–revisited

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**Abstract**

We consider the Hamilton-Jacobi equation

$$H(x, Du) + \lambda(x)u = c, \quad x \in M,$$

where $M$ is a connected, closed and smooth Riemannian manifold. The functions $H(x, p)$ and $\lambda(x)$ are continuous. $H(x, p)$ is convex, coercive with respect to $p$, and $\lambda(x)$ changes the signs. The first breakthrough to this model was achieved by Jin-Yan-Zhao [11] under the Tonelli conditions. In this paper, we consider more detailed structure of the viscosity solution set and large time behavior of the viscosity solution on the Cauchy problem. To the best of our knowledge, it is the first detailed description of the large time behavior of the HJ equations with non-monotone dependence on the unknown function.


*MSC:* 37J51; 35F21; 35D40

*Keywords:* Aubry-Mather theory; Weak KAM theory; Hamiltonian systems; Contact Hamiltonian systems

## Contents



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## 1. Introduction and main results

Let $H : T^*M \times \mathbb{R} \to \mathbb{R}$ be a contact Hamiltonian. It turns out that the dependence of $H$ on the contact variable $u$ plays a crucial role in exploiting the dynamics generated by $H$. By using previous dynamical approaches, some progress on viscosity solutions of Hamilton-Jacobi (HJ) equations have been achieved [16,17,19]. In particular, based on the works mentioned before, the structure of the set of solutions can be sketched if $H$ is uniformly Lipschitz in $u$. Shortly after [17] occurred, [12] generalized the results to ergodic problems by using PDE approaches. More recently, for a class of HJ equations with non-monotone dependence on $u$, the first breakthrough was achieved by Jin-Yan-Zhao [11] under the Tonelli conditions. In that work, they provided a description of the solution set of the stationary equation (formulated as ($E_0$) below) and revealed a bifurcation phenomenon with respect to the value $c$ in the right hand side, which opened a way to exploit further properties of viscosity solutions beyond well-posedness for HJ equations with non-monotone dependence on $u$. The main results in this paper are motivated by [11]. The present paper further discusses the large time behavior of the non-monotone model considered in [11]. To the best of our knowledge, Theorems 2 and 3 below are the first detailed description of the large time behavior of the HJ equations non-monotone in the unknown function. For another result on this topic, one can refer to [10, Theorem 6.5 (3)].

Let us consider the stationary equation:

$$H(x, Du) + \lambda(x)u = c, \quad x \in M. \tag{$E_0$}$$

Throughout this paper, we assume $M$ is a closed, connected and smooth Riemannian manifold. $D$ denotes the spacial gradient with respect to $x \in M$. Denote by $TM$ and $T^*M$ the tangent bundle and cotangent bundle of $M$ respectively. Let $H : T^*M \to \mathbb{R}$ satisfy

**(C):** $H(x, p)$ is continuous;
**(CON):** $H(x, p)$ is convex in $p$, for any $x \in M$;
**(CER):** $H(x, p)$ is coercive in $p$, i.e. $\lim_{\|p\|_x \to +\infty} H(x, p) = +\infty$, where $\|\cdot\|_x$ denotes the norms induced by $g$ on both $TM$ and $T^*M$.

Correspondingly, one has the Lagrangian associated to $H$:

$$L(x, \dot{x}) := \sup_{p \in T^*_x M} \{\langle \dot{x}, p\rangle_x - H(x, p)\},$$

where $\langle\cdot,\cdot\rangle_x$ represents the canonical pairing between $T_xM$ and $T_x^*M$. The Lagrangian $L(x,\dot{x})$ satisfies the following properties:

**(LSC):** $L(x,\dot{x})$ is lower semicontinuous in $\dot{x}$, and continuous on the interior of its domain $\mathrm{dom}(L):=\{(x,\dot{x})\in TM:\ L(x,\dot{x})<+\infty\}$;
**(CON):** $L(x,\dot{x})$ is convex in $\dot{x}$, for any $x\in M$.

We also assume $\lambda(x)$ is continuous and satisfies

**(±):** there exist $x_1,x_2\in M$ such that $\lambda(x_1)>0$ and $\lambda(x_2)<0$.

Throughout this paper, we define

$$\lambda_0:=\|\lambda(x)\|_\infty>0, \tag{1.1}$$

where $\|\cdot\|_\infty$ stands for the supremum norm of the functions on their domains. Based on this model, we revealed some different phenomena from the cases with monotone dependence on $u$ can be revealed.

**Remark 1.1.** The model ($E_0$) has been considered in [22]. In that paper, the function $\lambda(x)$ is non-negative and positive on the projected Aubry set of $H(x,p)$. In this case, the solution of ($E_0$) is unique. The asymptotic behavior of the solution of ($E_0$) is also studied in [22] when $\lambda_0\to 0^+$. When $\lambda_0\to 0^+$ and the assumption (±) holds, the family of solutions of ($E_0$) may diverge, one can refer to [13] for an example.

In [14], the well-posedness of the Lax-Oleinik semigroup was verified for contact HJ equations under very mild conditions. By virtue of that, we generalize the results in [11] to the cases from the Tonelli conditions to the assumptions (C), (CON) and (CER) above. Henceforth, for simplicity of notation, we omit the word “viscosity”, if it is not necessary to be mentioned.

**Proposition 1.2** *(Generalization of [11]). Let*

$$c_0:=\inf_{u\in C^\infty(M)}\sup_{x\in M}\Big\{H(x,Du)+\lambda(x)u\Big\}. \tag{1.2}$$

*Then $c_0$ is finite. Given $c\geq c_0$, the $\|\cdot\|_{W^{1,\infty}}$-norm of all subsolutions of ($E_0$) is bounded. Moreover,*

(1) *($E_0$) has a solution if and only if $c\geq c_0$;*
(2) *if $c>c_0$, then ($E_0$) has at least two solutions.*

The definition of $c_0$ is inspired by [4]. In light of that, $c_0$ is called the critical value. Now we consider the following case with a more general dependence of $H$ in $u$

$$H(x,u(x),Du(x))=c,\quad x\in M,$$

where the Hamiltonian $H(x,u,p)$ is continuous, superlinear in $p$ and uniformly Lipschitz in $u$. It was pointed out in [12] that there is a constant $c \in \mathbb{R}$ such that the above equation has viscosity solutions. Here we give some examples on the set $\mathfrak{C}$ of all such $c$, which reveal the essential differences between the monotone cases and the non-monotone cases:

- for classical Tonelli Hamiltonian $H(x,p)$, the set $\mathfrak{C} = \{c_0\}$. The number $c_0$ is called the Mañé critical value;
- for the discounted Hamilton-Jacobi equation, i.e., the Hamiltonian is of the form $\lambda u + H(x,p)$ with $\lambda > 0$, the set $\mathfrak{C} = \mathbb{R}$, see for example [6];
- for the model ($E_0$) considered here, the set $\mathfrak{C} = [c_0, +\infty)$. Here we note that the non-emptiness of $\mathfrak{C}$ is proved under (CER) instead of $H(x,p)$ is superlinear in $p$. In view of the existence result in [12], it means Proposition 1.2 is a non-trivial generalization of [11];
- for the Hamiltonian periodically depending on $u$, i.e., $H(x,u+1,p) \equiv H(x,u,p)$, the set $\mathfrak{C}$ is a bounded closed interval, see [15].

Different from the Tonelli case considered in [11], some new ingredients are needed for *a priori* estimates of subsolutions under the assumptions (C), (CON) and (CER). Those estimates will be provided in Section 3. The remaining parts of the proof of Proposition 1.2 are similar to the one in [11]. We postpone it to Appendix A.3 for consistency.

Motivated by Proposition 1.2, we are devoted to exploiting more detailed information of this model. First of all, we obtain

**Theorem 1.** *Let $c \geq c_0$. There exist the maximal element $u_{\max}$ and the minimal element $u_{\min}$ in the set of solutions of ($E_0$).*

**Remark 1.3.** The viscosity solutions are equivalent to backward weak KAM solutions in our setting (see [14, Proposition D.4]). In terms of the correspondence between backward and forward weak KAM solutions (see Proposition 2.8(3) below), it follows from Theorem 1 that there exist the maximal and minimal forward weak KAM solutions of ($E_0$). We denote $u^+_{\min}$ (resp. $u^+_{\max}$) the minimal (resp. maximal) froward weak KAM solution of ($E_0$). One can refer to Proposition 2.1 and (T-) below for the definition of the backward semigroup $T^-_t$ and the forward semigroup $T^+_t$. By Proposition 2.8(3)(4), there hold

$$u^+_{\min} \leq u_{\min} = \lim_{t\to+\infty} T^-_t u^+_{\min}, \quad \lim_{t\to+\infty} T^+_t u_{\max} = u^+_{\max} \leq u_{\max}.$$

Let $\mathcal{S}_-$ (resp. $\mathcal{S}_+$) be the set of all backward (resp. forward) weak KAM solutions. Given $u_\pm \in \mathcal{S}_\pm$, if

$$u_- = \lim_{t\to\infty} T^-_t u_+, \quad u_+ = \lim_{t\to\infty} T^+_t u_-,$$

then $u_-$ (resp. $u_+$) is called a conjugated backward (resp. forward) weak KAM solution. See Fig. 1 for a rough description of structure of the solution set of ($E_0$) in general cases, where $\mathbf{T}_\pm := \lim_{t\to\infty} T^\pm_t$, and $\mathcal{P}_-$ (resp. $\mathcal{P}_+$) denotes the set of all conjugated backward (resp. forward) weak KAM solutions. For further statement on conjugated weak KAM solutions, one can refer to [10, Theorem 6.5 and Theorem 7.1].

By Proposition 1.2(2), ($E_0$) has at least two solutions if $c > c_0$. Then a natural question is to figure out what happens if $c = c_0$. In [11], Jin, Yan and Zhao considered the following example:

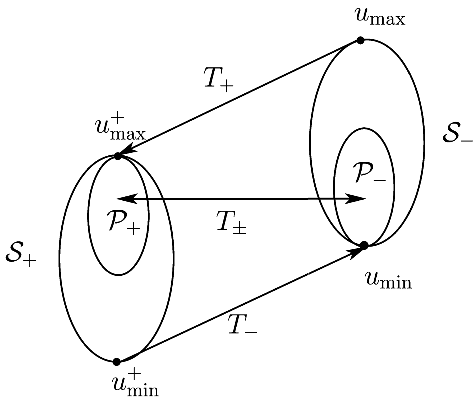


Fig. 1. The structure of the solution set of ($E_0$).

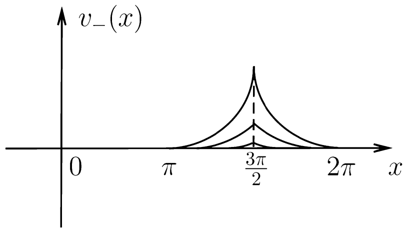


Fig. 2. Certain solutions of (1.3) with $c = 0$.

**Example 1.4.**

$$|u'(x)|^2 + \sin x \cdot u(x) = c, \quad x \in \mathbb{S}^1 \simeq [0, 2\pi), \tag{1.3}$$

where $\mathbb{S}^1$ denotes a flat circle with a fundamental domain $[0, 2\pi)$.

It was shown that $c_0 = 0$ and there are uncountably many solutions of (1.3) in the critical case. A rough picture of certain solutions is given by Fig. 2. See [11, Theorem 3.5] for more details.

As a complement, we consider

**Example 1.5.**

$$\frac{1}{2}|u'(x)|^2 + \sin x \cdot u(x) + \cos 2x - 1 = c, \quad x \in \mathbb{S}^1 \simeq [0, 2\pi). \tag{1.4}$$

We will prove that the critical value is also $c_0 = 0$, but (1.4) admits a unique solution in the critical case. A rough picture of the solution is given by Fig. 3. See Remark 4.2 below for certain generalization of Example 1.5. Those two examples above show various possibilities about the solution set of ($E_0$) in the critical case.

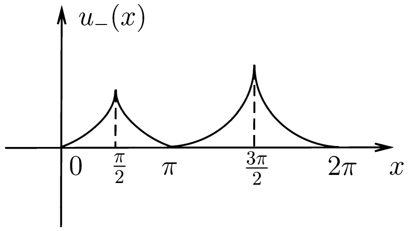


Fig. 3. The unique solution of (1.4) with $c = 0$.

In the second part, we consider the evolutionary equation:

$$\begin{cases} \partial_t u(x,t) + H(x, Du(x,t)) + \lambda(x)u(x,t) = c, & (x,t) \in M \times (0,+\infty). \\ u(x,0) = \varphi(x), & x \in M, \end{cases} \tag{CP}$$

where $\varphi \in C(M)$. It is well known that the viscosity solution of (CP) is unique (see [10, Corollary 3.2] for instance). By [14, Theorem 1], this solution can be represented by $u(x,t) := T_t^-\varphi(x)$, where $T_t^- : C(M) \to C(M)$ is defined implicitly by

$$T_t^-\varphi(x) = \inf_{\gamma(t)=x} \left\{ \varphi(\gamma(0)) + \int_0^t \left[ L(\gamma(\tau), \dot\gamma(\tau)) - \lambda(\gamma(\tau)) T_\tau^-\varphi(\gamma(\tau)) + c \right] d\tau \right\}, \tag{T-}$$

where the infimum is taken among absolutely continuous curves $\gamma : [0,t] \to M$ with $\gamma(t) = x$.

In order to obtain equi-Lipschitz continuity of $\{T_t^-\varphi\}_{t\geq\delta}$ for a given $\delta > 0$, we have to strengthen the assumptions on $H$ from (CON), (CER) to the following:

($\star$) $H(x,p)$ is strictly convex in $p$ for any $x \in M$, and there is a superlinear function $\theta : [0,+\infty) \to [0,+\infty)$ such that $H(x,p) \geq \theta(\|p\|)$.

Under the assumption ($\star$), the equi-Lipschitz continuity of $\{T_t^-\varphi\}_{t\geq\delta}$ follows from the locally Lipschitz property and boundedness of $T_t^-\varphi$ on $M \times (0,+\infty)$. From the weak KAM point of view, that kind of locally Lipschitz property can be verified by a standard procedure once we have the Lipschitz regularity of minimizers of $T_t^-\varphi(x)$ (see [7, Lemma 4.6.3]). However, $H$ is only supposed to be continuous in our setting. Then one can not use the method of characteristics to improve regularity of these minimizers. Following [1], we will deal with that issue by using the method of energy estimates. A key ingredient of that method is to establish the Erdmann condition for a non-smooth energy function. More precisely, we obtain the following result, whose proof is given in Appendix A.4.

**Proposition 1.6.** *Assume ($\star$) holds. If $T_t^-\varphi(x)$ has a bound independent of $t$, then the family $\{T_t^-\varphi\}_{t\geq\delta}$ is equi-Lipschitz continuous, where $\delta$ is an arbitrarily positive constant.*

Let us recall $u_{\max}$ denotes the maximal solution of ($E_0$), and $u^+_{\min}$ denotes its minimal froward weak KAM solution. By Remark 1.3, $u^+_{\min} \leq u_{\max}$ on $M$. Both of them play important roles in characterizing the large time behavior of the solution of (CP). By assuming ($\star$) holds, we obtain the following two results.

**Theorem 2.** *Let $u(x,t)$ be the solution of (CP) with $c \geq c_0$. Then*

(1) *if the initial data $\varphi \geq u_{\max}$, then $u(x,t)$ converges to $u_{\max}$ uniformly on $M$ as $t \to +\infty$;*
(2) *if there is a point $x_0 \in M$ such that $\varphi(x_0) < u^+_{\min}(x_0)$, then $u(x,t)$ tends to $-\infty$ uniformly on $M$ as $t \to +\infty$.*

**Theorem 3.** *Let $u(x,t)$ be the solution of (CP) with $c > c_0$. If the initial data $\varphi > u^+_{\min}$, then $u(x,t)$ converges to $u_{\max}$ uniformly on $M$ as $t \to +\infty$.*

**Remark 1.7.** For $\varphi \geq u^+_{\min}$, if there exists $x_0 \in M$ such that $\varphi(x_0) = u^+_{\min}(x_0)$, then $u(x,t)$ may not converge to $u_{\max}$.

- In Example 1.4 with $c = c_0$, for each solution $v$ of (1.3), it is easy to construct an initial data $\varphi$ satisfying $\varphi \geq 0 \geq u^+_{\min}$ and

$$\{x \in M \mid \varphi(x) = u^+_{\min}(x)\} \neq \emptyset$$

such that $u(x,t)$ converges to $v$ uniformly on $M$. In fact, one can take $\varphi = v$ for instance.
- For Example 1.4 with $c = 1 > c_0$, by [11, Theorem 3.14], $u_{\min} = \sin x \neq u_{\max}$ and

$$\{x \in M \mid u_{\min}(x) = u^+_{\min}(x)\} \neq \emptyset.$$

Then one can take $\varphi = \sin x$ such that $u(x,t)$ converges to $u_{\min}$ uniformly on $M$.
- More exotically, $u(x,t)$ may converge (up to a subsequence) uniformly to a nontrivial time periodic solution of

$$\partial_t u(x,t) + H(x, Du(x,t)) + \lambda(x)u(x,t) = c.$$

Inspired by [21, Example 5.1] and [20, Theorem 1.5], we consider evolutionary the HJ equation:

$$\begin{cases} \partial_t u(x,t) + \dfrac{1}{2}|Du(x,t)|^2 + Du(x,t) + \left(\sin 2\pi x - \dfrac{1}{2}\right) u(x,t) = 0, \\ u(x,0) = \varphi(x), \quad x \in \mathbb{S}^1 \simeq [0,1), \end{cases} \tag{CP$_e$}$$

and its associated stationary equation:

$$\frac{1}{2}|Du|^2 + Du + \left(\sin 2\pi x - \frac{1}{2}\right) u = 0. \tag{S$_e$}$$

It is clear $u_0 \equiv 0$ is a solution of (S$_e$). According to [21, Lemma 2.2(1)], there exists a strict subsolution $v(x)$ with $v > 0$ on $\mathbb{S}^1$. It implies the critical value (defined by (1.2)) $c_0 < 0$. By Lemma 5.4 below, $u_{\max} = \lim_{t\to+\infty} T^-_t v > v > 0$ and $u^+_{\min} = \lim_{t\to+\infty} T^+_t v$. Since $v > u_0 \equiv 0$, we get $T^+_t v > T^+_t u_0 \equiv 0$. Then $u^+_{\min} \geq 0$. Since $u_0 \equiv 0$ is a classical solution of (S$_e$), $u^+_{\min} \leq u_0 \equiv 0$. We conclude that $u^+_{\min} \equiv 0$. Now let $\varphi \geq 0$ be a non-vanishing initial data $\varphi \geq 0$ satisfying

$$\{x \in \mathbb{S}^1 \mid \varphi(x) = 0\} \neq \emptyset.$$

Similar to [20, Theorem 1.5], we can prove that $w_\varphi(x,t) := \lim_{n\to+\infty} T^-_{n+t}\varphi$ exists for $n \in \mathbb{N}$. In light of [21, Theorem 1.4], $w_\varphi(x,t)$ is a nontrivial time periodic solution of

$$\partial_t u(x,t) + \frac{1}{2}|Du(x,t)|^2 + Du(x,t) + \left(\sin 2\pi x - \frac{1}{2}\right) u(x,t) = 0.$$

**Remark 1.8.** Now we recall the previous results on the large time behavior of the HJ equations monotone in the unknown function. Consider the evolutionary equation:

$$\begin{cases} \partial_t u(x,t) + H(x, u(x,t), Du(x,t)) = 0, & (x,t) \in M \times (0,+\infty). \\ u(x,0) = \varphi(x), & x \in M, \end{cases} \tag{A$_0$}$$

and the stationary equation:

$$H(x, u(x), Du(x)) = 0 \tag{B$_0$}$$

(a) When the Hamiltonian is increasing in the unknown function, according to [16, Theorem 1.4], the solution of ($A_0$) uniformly converges to a solution of ($B_0$) as $t \to +\infty$ for each initial data $\varphi(x)$.
(b) When the Hamiltonian is strictly decreasing in the unknown function, according to [19, Theorem 2], if $\varphi > u_+$, then the solution of ($A_0$) uniformly converges to $+\infty$ as $t \to +\infty$. If there is a point $x_0$ such that $\varphi(x_0) < u_+(x_0)$, then the solution of ($A_0$) uniformly converges to $-\infty$ as $t \to +\infty$. Here $u_+$ is the unique forward weak KAM solution of ($B_0$).

From the results above, we can see that the non-monotone model (CP) has both characteristics of Case (a) and Case (b).

The rest of this paper is organized as follows. Section 2 gives some preliminaries on $T_t^\pm$, weak KAM solutions and Aubry sets. In Section 3, *a priori* estimates on subsolutions of ($E_0$) are established. The proof of Theorem 1 and a detailed analysis of Example 1.5 are given in Section 4. Theorem 2 and Theorem 3 are proved in Section 5. For the sake of completeness, some auxiliary results are proved in Appendix A.

## 2. Preliminaries

In this part, we collect some facts on $T_t^\pm$, weak KAM solutions and Aubry sets. These facts hold under more general assumptions on the dependence of $u$. We denote by $(x,u,p)$ a point in $T^*M \times \mathbb{R}$, where $(x,p) \in T^*M$ and $u \in \mathbb{R}$. Let $H : T^*M \times \mathbb{R} \to \mathbb{R}$ be a continuous Hamiltonian satisfying

**(CON):** $H(x,u,p)$ is convex in $p$, for any $(x,u) \in M \times \mathbb{R}$;
**(CER):** $H(x,u,p)$ is coercive in $p$, i.e. $\lim_{\|p\|_x\to+\infty}(\inf_{x\in M} H(x,0,p)) = +\infty$;
**(LIP):** $H(x,u,p)$ is Lipschitz in $u$, uniformly with respect to $(x,p)$, i.e., there exists $\Theta > 0$ such that $|H(x,u,p) - H(x,v,p)| \leq \Theta|u-v|$, for all $(x,p) \in T^*M$ and all $u, v \in \mathbb{R}$.

Correspondingly, one has the Lagrangian associated to $H$:

$$L(x,u,\dot{x}) := \sup_{p\in T_x^*M} \{\langle \dot{x}, p\rangle_x - H(x,u,p)\}.$$

Due to the absence of superlinearity of $H$, the corresponding Lagrangian $L$ may take the value $+\infty$. Define

$$\mathrm{dom}(L) := \{(x,\dot{x},u) \in TM\times\mathbb{R} \mid L(x,u,\dot{x}) < +\infty\}.$$

By the Lipschitz dependence of $L$ in $u$, we have (see [14, Remark 1.2])

$$\mathrm{dom}(L) = \{(x,\dot{x}) \in TM \mid L(x,0,\dot{x}) < +\infty\} \times \mathbb{R}.$$

Then $L(x,u,\dot{x})$ satisfies the following properties:

**(LSC):** $L(x,u,\dot{x})$ is lower semicontinuous, and continuous on the interior of $\mathrm{dom}(L)$;
**(CON):** $L(x,u,\dot{x})$ is convex in $\dot{x}$, for any $(x,u) \in M\times\mathbb{R}$;
**(LIP):** $L(x,u,\dot{x})$ is Lipschitz in $u$, uniformly with respect to $(x,\dot{x})$, i.e., there exists $\Theta > 0$ such that $|L(x,u,\dot{x}) - L(x,v,\dot{x})| \le \Theta|u-v|$, for all $(x,\dot{x},u)$ and $(x,\dot{x},v) \in \mathrm{dom}(L)$.

Here (LSC) follows from basic facts of convex analysis (see [5, Theorem A.3]).

**Proposition 2.1.** *[14, Theorem 1] Both the backward Lax-Oleinik semigroup*

$$T_t^-\varphi(x) = \inf_{\gamma(t)=x}\left\{\varphi(\gamma(0)) + \int_0^t L(\gamma(\tau), T_\tau^-\varphi(\gamma(\tau)), \dot{\gamma}(\tau))d\tau\right\} \tag{2.1}$$

*and the forward Lax-Oleinik semigroup*

$$T_t^+\varphi(x) = \sup_{\gamma(0)=x}\left\{\varphi(\gamma(t)) - \int_0^t L(\gamma(\tau), T_{t-\tau}^+\varphi(\gamma(\tau)), \dot{\gamma}(\tau))d\tau\right\}, \tag{2.2}$$

*are well-defined for $\varphi \in C(M)$. The infimum (resp. supremum) is taken among absolutely continuous curves $\gamma : [0,t] \to M$ with $\gamma(t) = x$ (resp. $\gamma(0) = x$). If $\varphi$ is continuous, then $u(x,t) := T_t^-\varphi(x)$ represents the unique continuous viscosity solution of ($A_0$). If $\varphi$ is Lipschitz continuous, then $u(x,t) := T_t^-\varphi(x)$ is also locally Lipschitz continuous on $M\times[0,+\infty)$.*

**Proposition 2.2.** *[14, Proposition 3.1] The Lax-Oleinik semigroups have the following properties*

(1) *For $\varphi_1$ and $\varphi_2 \in C(M)$, if $\varphi_1(x) < \varphi_2(x)$ for all $x \in M$, we have $T_t^-\varphi_1(x) < T_t^-\varphi_2(x)$ and $T_t^+\varphi_1(x) < T_t^+\varphi_2(x)$ for all $(x,t) \in M\times(0,+\infty)$.*
(2) *Given any $\varphi$ and $\psi \in C(M)$, we have $\|T_t^-\varphi - T_t^-\psi\|_\infty \le e^{\Theta t}\|\varphi-\psi\|_\infty$ and $\|T_t^+\varphi - T_t^+\psi\|_\infty \le e^{\Theta t}\|\varphi-\psi\|_\infty$ for all $t > 0$.*

Following Fathi [7], one can extend the definitions of backward and forward weak KAM solutions of equation ($B_0$) by using absolutely continuous calibrated curves instead of $C^1$ curves.

**Definition 2.3.** A function $u_- \in C(M)$ is called a backward weak KAM solution of ($B_0$) if

(1) For each absolutely continuous curve $\gamma : [t', t] \to M$, we have

$$u_-(\gamma(t)) - u_-(\gamma(t')) \leq \int_{t'}^{t} L(\gamma(s), u_-(\gamma(s)), \dot{\gamma}(s))ds.$$

The above condition reads that $u_-$ is dominated by $L$ and denoted by $u_- \prec L$.

(2) For each $x \in M$, there exists an absolutely continuous curve $\gamma_- : (-\infty, 0] \to M$ with $\gamma_-(0) = x$ such that

$$u_-(x) - u_-(\gamma_-(t)) = \int_{t}^{0} L(\gamma_-(s), u_-(\gamma_-(s)), \dot{\gamma}_-(s))ds, \quad \forall t < 0.$$

The curves satisfying the above equality are called $(u_-, L, 0)$-calibrated curves.

A forward weak KAM solution of ($B_0$) can be defined in a similar manner. Similar to [18, Proposition 2.8], one has

**Proposition 2.4.** *Let $\varphi \in C(M)$. Then*

$$-T_t^+(-\varphi) = \bar{T}_t^-\varphi, \quad -T_t^-(-\varphi) = \bar{T}_t^+\varphi, \quad \forall t \geq 0, \tag{2.3}$$

*where $\bar{T}_t^\pm$ denote the Lax-Oleinik semigroups associated to $L(x, -u, -\dot{x})$.*

The following two results are well known for Hamilton-Jacobi equations independent of $u$. They are also true in contact cases. We will prove them in Appendices A.1 and A.2. Proposition 2.5 provides some equivalent characterizations of Lipschitz subsolutions. Proposition 2.6 shows that $T_t^+$ is a 'weak inverse' of $T_t^-$.

**Proposition 2.5.** *Let $\varphi \in Lip(M)$. The following conditions are equivalent:*

(1) *$\varphi$ is a Lipschitz subsolution of ($B_0$);*
(2) *$\varphi \prec L$;*
(3) *for each $t \geq 0$,*

$$T_t^-\varphi \geq \varphi \geq T_t^+\varphi.$$

**Proposition 2.6.** *For each $\varphi \in C(M)$, we have $T_t^+ \circ T_t^-\varphi \leq \varphi \leq T_t^- \circ T_t^+\varphi$ for all $t \geq 0$.*

The following three results come from [14], which give some connections among the fixed points of $T_t^{\pm}$, the lower (resp. upper) half limit, backward (resp. forward) weak KAM solutions and Aubry sets.

**Proposition 2.7.** *[14, Proposition D.4] Let $u_- \in C(M)$. The following statements are equivalent:*

(1) *$u_-$ is a fixed point of $T_t^-$;*
(2) *$u_-$ is a backward weak KAM solution of ($B_0$);*
(3) *$u_-$ is a viscosity solution of ($B_0$).*

*Similarly, let $v_+ \in C(M)$. The following statements are equivalent:*

(1') *$v_+$ is a fixed point of $T_t^+$;*
(2') *$v_+$ is a forward weak KAM solution of ($B_0$);*
(3)' *$-v_+$ is a viscosity solution of $H(x, -u(x), -Du(x)) = 0$.*

**Proposition 2.8.** *[14, Theorem 3 and Remark 3.5] Let $\varphi \in C(M)$.*

(1) *If $T_t^-\varphi(x)$ has a bound independent of $t$, then the lower half limit*

$$\check{\varphi}(x) = \lim_{r\to 0+} \inf\{T_t^-\varphi(y) : d(x, y) < r,\ t > 1/r\}$$

*is a Lipschitz solution of ($B_0$).*

(2) *If $T_t^+\varphi(x)$ has a bound independent of $t$, then the upper half limit*

$$\hat{\varphi}(x) = \lim_{r\to 0+} \sup\{T_t^+\varphi(y) : d(x, y) < r,\ t > 1/r\},$$

*is a Lipschitz forward weak KAM solution of ($B_0$).*

(3) *Let $u_-$ be a solution of ($B_0$). Then $T_t^+ u_- \le u_-$. The limit $u_+ := \lim_{t\to+\infty} T_t^+ u_-$ exists, and $u_+$ is a forward weak KAM solution of ($B_0$).*
(4) *Let $v_+$ be a forward weak KAM solution of ($B_0$). Then $T_t^- v_+ \ge v_+$. The limit $v_- := \lim_{t\to+\infty} T_t^- v_+$ exists, and $v_-$ is a solution of ($B_0$).*

**Proposition 2.9.** *[14, Theorem 3] Let $u_-$ (resp. $u_+$) be a solution (resp. forward weak KAM solution) of ($B_0$). We define the projected Aubry set with respect to $u_-$ by*

$$\mathcal{I}_{u_-} := \{x \in M : u_-(x) = \lim_{t\to+\infty} T_t^+ u_-(x)\}.$$

*Correspondingly, we define the projected Aubry set with respect to $u_+$ by*

$$\mathcal{I}_{u_+} := \{x \in M : u_+(x) = \lim_{t\to+\infty} T_t^- u_+(x)\}.$$

*Both $\mathcal{I}_{u_-}$ and $\mathcal{I}_{u_+}$ are nonempty. In particular, if $u_+(x) = \lim_{t\to+\infty} T_t^+ u_-(x)$ and $u_-(x) = \lim_{t\to+\infty} T_t^- u_+(x)$, then*

$$\mathcal{I}_{u_-} = \mathcal{I}_{u_+},$$

*which is also denoted by $\mathcal{I}_{(u_-,u_+)}$, following the notation introduced by Fathi [7].*

## 3. Some estimates on subsolutions

In this section, we assume the existence of subsolutions of ($E_0$) and prove some *a priori* estimates on subsolutions. The existence of subsolutions will be verified for $c \geq c_0$ in Proposition A.6 below.

Let $L(x, \dot{x})$ be the Lagrangian associated to $H(x, p)$. Let $T_t^{\pm}$ be the Lax-Oleinik semigroups associated to

$$L(x, \dot{x}) - \lambda(x)u(x) + c.$$

Similar to [9, Proposition 2.1], one can prove the local boundedness of $L(x, \dot{x})$ in a neighborhood of the zero section of $TM$.

**Lemma 3.1.** *Let $H(x, p)$ satisfy (C)(CON)(CER), there exist constants $\delta > 0$ and $C_L > 0$ such that the Lagrangian $L(x, \dot{x})$ associated to $H(x, p)$ satisfies*

$$L(x, \xi) \leq C_L, \quad \forall (x, \xi) \in M \times \bar{B}(0, \delta). \tag{3.1}$$

Throughout this paper, we define

$$\mu := \mathrm{diam}(M)/\delta, \tag{3.2}$$

where $\mathrm{diam}(M)$ denotes the diameter of $M$.

**Lemma 3.2.** *Let $\varphi \in C(M)$. Then*

(1) *$T_t^- \varphi$ has an upper bound independent of $t$;*
(2) *$T_t^+ \varphi$ has a lower bound independent of $t$.*

**Proof.** Taking $x_1 \in M$ with $\lambda(x_1) > 0$. We first show

$$T_t^- \varphi(x_1) \leq \max\left\{\varphi(x_1), \frac{L(x_1, 0) + c}{\lambda(x_1)}\right\}, \quad \forall t \geq 0.$$

Otherwise, there is $t > 0$ such that

$$T_t^- \varphi(x_1) > \max\left\{\varphi(x_1), \frac{L(x_1, 0) + c}{\lambda(x_1)}\right\} \geq \frac{L(x_1, 0) + c}{\lambda(x_1)}.$$

There are two cases:

(i) For all $s \in [0, t]$, we have

$$T_s^- \varphi(x_1) > \frac{L(x_1, 0) + c}{\lambda(x_1)}.$$

Taking the constant curve $\gamma \equiv x_1$, we have

$$T_t^- \varphi(x_1) \leq \varphi(x_1) + \int_0^t \Big[ L(x_1, 0) + c - \lambda(x_1) T_s^- \varphi(x_1) \Big] ds < \varphi(x_1),$$

which also leads to a contradiction.

(ii) There is $t_0 \geq 0$ such that

$$T_{t_0}^- \varphi(x_1) = \frac{L(x_1, 0) + c}{\lambda(x_1)},$$

and

$$T_s^- \varphi(x_1) > \frac{L(x_1, 0) + c}{\lambda(x_1)}, \quad \forall s \in (t_0, t].$$

Taking the constant curve $\gamma \equiv x_1$, we have

$$T_t^- \varphi(x_1) \leq T_{t_0}^- \varphi(x_1) + \int_0^t \Big[ L(x_1, 0) + c - \lambda(x_1) T_s^- \varphi(x_1) \Big] ds < \frac{L(x_1, 0) + c}{\lambda(x_1)},$$

which leads to a contradiction.

We then prove that for all $x \in M$ and all $t > 0$, $T_t^- \varphi(x)$ is bounded from above. It suffices to prove that for all $x \in M$ and $t > 0$, $T_{t+\mu}^- \varphi(x)$ is bounded from above, where $\mu$ is given by (3.2). Let $\alpha : [0, \mu] \to M$ be a geodesic connecting $x_1$ and $x$ with constant speed, then $\|\dot{\alpha}\| \leq \delta$. Let

$$K_0 := \max \left\{ \varphi(x_1), \frac{L(x_1, 0) + c}{\lambda(x_1)} \right\}.$$

Given $x \neq x_1$. We assume $T_{t+\mu}^- \varphi(x) > K_0$. Otherwise the proof is completed. Since $T_t^- \varphi(x_1) \leq K_0$, there exists $\sigma \in [0, \mu)$ such that $T_{t+\sigma}^- \varphi(\alpha(\sigma)) = K_0$ and $T_{t+s}^- \varphi(\alpha(s)) > K_0$ for all $s \in (\sigma, \mu]$. By definition

$$\begin{aligned} T_{t+s}^- \varphi(\alpha(s)) &\leq T_{t+\sigma}^- \varphi(\alpha(\sigma)) + \int_\sigma^s \Big[ L(\alpha(\tau), \dot{\alpha}(\tau)) - \lambda(\alpha(\tau)) \cdot T_{t+\tau}^- \varphi(\alpha(\tau)) + c \Big] d\tau \\ &= K_0 + \int_\sigma^s \Big[ L(\alpha(\tau), \dot{\alpha}(\tau)) - \lambda(\alpha(\tau)) \cdot T_{t+\tau}^- \varphi(\alpha(\tau)) + c \Big] d\tau, \end{aligned}$$

which implies

$$
\begin{aligned}
T^-_{t+s}\varphi(\alpha(s)) - K_0 &\le \int_\sigma^s \Big[L(\alpha(\tau),\dot\alpha(\tau)) - \lambda(\alpha(\tau))\cdot T^-_{t+\tau}\varphi(\alpha(\tau)) + c\Big]d\tau \\
&\le \int_\sigma^s \Big[L(\alpha(\tau),\dot\alpha(\tau)) - \lambda(\alpha(\tau))\cdot K_0 + c\Big]d\tau + \lambda_0\int_\sigma^s \Big[T^-_{t+\tau}\varphi(\alpha(\tau)) - K_0\Big]d\tau \\
&\le L_0\mu + \lambda_0\int_\sigma^s \Big[T^-_{t+\tau}\varphi(\alpha(\tau)) - K_0\Big]d\tau,
\end{aligned}
$$

where $\lambda_0$ is given by (1.1) and

$$L_0 := C_L + \lambda_0 K_0 + c,$$

where $C_L$ is given by (3.1). By the Gronwall inequality, we have

$$T^-_{t+s}\varphi(\alpha(s)) - K_0 \le L_0\mu e^{\lambda_0(s-\sigma)} \le L_0\mu e^{\lambda_0\mu}, \quad \forall s\in(\sigma,\mu].$$

Taking $s=\mu$ we have $T^-_{t+\mu}\varphi(x) \le K_0 + L_0\mu e^{\lambda_0\mu}$.

Similar to the argument above, by choosing constant curve $\gamma(\tau)\equiv x_2$ with $\tau\in[0,t]$ and replacing $T^-_{t+\mu}\varphi$ by $T^+_{t+\mu}\varphi$, one has

$$T^+_t\varphi(x) \ge \min\left\{\varphi(x_2), \frac{L(x_2,0)+c}{\lambda(x_2)}\right\} - L_0\mu e^{\lambda_0\mu}. \tag{3.3}$$

This completes the proof. □

**Corollary 3.3.** *Let $u_0$ be a Lipschitz subsolution of ($E_0$). Then $T^-_t u_0$ (resp. $T^+_t u_0$) has an upper (resp. lower) bound independent of $t$ and $u_0$.*

**Proof.** We only prove $T^-_t u_0$ has an upper bound independent of $t$ and $u_0$. The case with $T^+_t u_0$ is similar. Let

$$\mathbf{e}_0 := \min_{(x,p)\in T^*M} H(x,p). \tag{3.4}$$

By (CER), $\mathbf{e}_0$ is finite. By the definition of the subsolution, $H(x_1,p)+\lambda(x_1)u_0(x_1)\leqslant c$ for any $p\in D^*u_0(x_1)$, where $D^*$ denotes the reachable gradients. It implies

$$\lambda(x_1)u_0(x_1) \leqslant c - \min_{(x,p)\in T^*M} H(x,p) = c - \mathbf{e}_0.$$

Hence, for each subsolution $u_0$, we have

$$u_0(x_1) \le \frac{c-\mathbf{e}_0}{\lambda(x_1)}.$$

Let

$$K_0 := \frac{c - \mathbf{e}_0}{\lambda(x_1)}, \quad L_0 := C_L + \lambda_0 K_0 + c,$$

where $\lambda_0$ is given by (1.1). Here we note that

$$L(x_1, 0) + c = \sup_{p \in T_x^* M} (-H(x_1, p)) + c \le - \min_{(x,p) \in T^* M} H(x, p) + c = c - \mathbf{e}_0.$$

By Lemma 3.2, we have

$$T_t^- u_0(x) \leqslant K_0 + L_0 \mu e^{\lambda_0 \mu}. \tag{3.5}$$

This completes the proof. □

**Proposition 3.4.** *There exists a constant $C > 0$ such that for any subsolution $u$ of ($E_0$), there holds*

$$\|u\|_{W^{1,\infty}} \leqslant C.$$

**Proof.** By Proposition 2.5, for each $t \ge 0$,

$$T_t^+ u \le u \le T_t^- u.$$

By Corollary 3.3, there exist $C_1$, $C_2$ independent of $u$ such that

$$C_2 \le u \le C_1.$$

For each $x, y \in M$, let $\alpha : [0, d(x, y)/\delta] \to M$ be a geodesic of length $d(x, y)$ with constant speed $\|\dot{\alpha}\| = \delta$ and connecting $x$ and $y$, where $d(x, y)$ denotes the distance between $x$ and $y$ induced by the Riemannian metric $g$ on $M$. Then

$$L(\alpha(s), \dot{\alpha}(s)) \le C_L, \quad \forall s \in [0, d(x, y)/\delta].$$

By Proposition 2.5,

$$\begin{aligned} u(y) - u(x) &\le \int_0^{d(x,y)/\delta} \Big[ L(\alpha(s), \dot{\alpha}(s)) - \lambda(\alpha(s)) u(\alpha(s)) + c \Big] ds \\ &\le \frac{1}{\delta} \Big( C_L + \lambda_0 \max\{|C_1|, |C_2|\} + c \Big) d(x, y) =: \kappa d(x, y). \end{aligned}$$

Note that $\kappa$ is independent of the choice of the subsolution $u$. We get the equi-Lipschitz continuity of $u$ by exchanging the role of $x$ and $y$. □

**Proposition 3.5.** *Let $u_0$ be a Lipschitz subsolution of (E$_0$). Then*

$$u_- := \lim_{t\to+\infty} T_t^- u_0(x), \quad u_+ := \lim_{t\to+\infty} T_t^+ u_0(x)$$

*exist, and the limit procedure is uniform in $x$. Moreover, $u_-$ is a solution of (E$_0$), and $u_+$ is a forward weak KAM solution of (E$_0$). In particular, (E$_0$) has a solution $u_-$ for $c \geq c_0$.*

**Proof.** We only prove that $u_- := \lim_{t\to+\infty} T_t^- u_0(x)$ exists, and it is a viscosity solution of (E$_0$). The existence of $u_+$ is similar. By Proposition 2.8

$$\check{u}_-(x) := \lim_{r\to 0+} \inf\{T_t^- u_0(y) : \ d(x, y) < r, \ t > 1/r\}$$

is a solution of (E$_0$). By Proposition 2.5(3) and Corollary 3.3, for a given $x \in M$, the limit $\lim_{t\to+\infty} T_t^- u_0(x)$ exists. By definition, we have

$$\check{u}_-(x) \leq \lim_{t\to+\infty} T_t^- u_0(x).$$

Using Proposition 2.5(3) again, $T_t^- u_0$ is increasing in $t$ for all $t > 0$, we have

$$\begin{aligned} T_t^- u_0(x) &= \lim_{r\to 0+} \inf\{T_t^- u_0(y) : \ d(x, y) < r\} \\ &\leq \lim_{r\to 0+} \inf\{T_{t+s}^- u_0(y) : \ d(x, y) < r, \ t + s > 1/r\} = \check{u}_-(x). \end{aligned}$$

Then $\lim_{t\to+\infty} T_t^- u_0 = \check{u}_-$. Note that $\check{u}_-$ is a solution of (E$_0$). By Dini's theorem, the family $\{T_t^- u_0\}_{t>0}$ uniformly converges to $\check{u}_-$. □

## 4. Structure of the solution set of (E$_0$)

Let $\mathcal{S}_-$ (resp. $\mathcal{S}_+$) be the set of all solutions (resp. forward weak KAM solution) of (E$_0$).

### *4.1. The maximal solution*

We first prove the existence of the maximal solution. Since each solution is a subsolution, by Proposition 3.4, there are $C_1$ and $C_2$ such that $C_2 \leq u_- \leq C_1$ for all $u_- \in \mathcal{S}_-$. Note that all solutions of (E$_0$) are fixed points of $T_t^-$. We take a continuous function $\varphi > C_1$ as the initial data. By Proposition 2.2 (1), $T_t^- \varphi$ is larger than every solution of (E$_0$). By Lemma 3.2(1), $T_t^- \varphi$ has an upper bound independent of $t$. By Proposition 2.8 (1), the lower half limit

$$\check{\varphi}(x) = \lim_{r\to 0+} \inf\{T_t^- \varphi(y) : \ d(x, y) < r, \ t > 1/r\}$$

is a Lipschitz continuous viscosity solution of (E$_0$). Since $T_t^- \varphi$ is larger than every solution of (E$_0$), we have

$$\begin{aligned}\check{\varphi}(x) &= \lim_{r\to 0+} \inf\{T_t^- \varphi(y) : d(x,y) < r,\ t > 1/r\} \\ &\geq \lim_{r\to 0+} \inf\{v_-(y) : d(x,y) < r\} = v_-(x),\end{aligned}$$

for all $v_- \in \mathcal{S}_-$. Thus, $\check{\varphi}(x)$ is the maximal solution of (E$_0$).

### 4.2. The minimal solution

Since each forward weak KAM solution is dominated by $L(x,\dot{x}) - \lambda(x)u + c$, by Proposition 2.7, it is a subsolution of (E$_0$). By Proposition 3.4, there are $C_1$ and $C_2$ such that $C_2 \leq u_+ \leq C_1$ for all $u_+ \in \mathcal{S}_+$. We take a continuous function $\varphi < C_2$ as the initial data. By Proposition 2.2 (1), $T_t^+\varphi$ is smaller than every forward weak KAM solution of (E$_0$). By Lemma 3.2(2), $T_t^+\varphi$ has a lower bound independent of $t$. By Proposition 2.8 (2), the upper half limit

$$\hat{\varphi}(x) = \lim_{r\to 0+} \sup\{T_t^+ \varphi(y) : d(x,y) < r,\ t > 1/r\}$$

is a forward weak KAM solution of (E$_0$). Since $T_t^+\varphi$ is smaller than every forward weak KAM solutions of (E$_0$), we have

$$\begin{aligned}\hat{\varphi}(x) &= \lim_{r\to 0+} \sup\{T_t^+ \varphi(y) : d(x,y) < r,\ t > 1/r\} \\ &\leq \lim_{r\to 0+} \sup\{v_+(y) : d(x,y) < r\} = v_+(x),\end{aligned}$$

for all $v_+ \in \mathcal{S}_+$. Thus, $\hat{\varphi}(x)$ is the minimal forward weak KAM solution of (E$_0$). By Proposition 2.8 (4), $\hat{\varphi}_\infty := \lim_{t\to+\infty} T_t^- \hat{\varphi}$ exists, and it is a solution of (E$_0$).

**Lemma 4.1.** *$\hat{\varphi}_\infty$ is the minimal solution of (E$_0$).*

**Proof.** Define

$$\mathcal{P}_- := \{u_- \in \mathcal{S}_- : \exists u_+ \in \mathcal{S}_+ \text{ such that } u_- = \lim_{t\to+\infty} T_t^- u_+\}.$$

We first prove that for each $v_- \in \mathcal{P}_-$, there holds $v_- \geq \hat{\varphi}_\infty$. In fact, by definition of $\mathcal{P}_-$, there is $u_+ \in \mathcal{S}_+$ such that $v_- = \lim_{t\to+\infty} T_t^- u_+$. Since $\hat{\varphi}$ is the minimal forward weak KAM solution, we have

$$u_+ \geq \hat{\varphi}.$$

Acting $T_t^-$ on both sides of the inequality above, and letting $t \to +\infty$, we have $v_- \geq \hat{\varphi}_\infty$.

We then prove that for each $v_- \in \mathcal{S}_- \backslash \mathcal{P}_-$, $v_- \geq \hat{\varphi}_\infty$ still holds. Let $v_+ := \lim_{t\to+\infty} T_t^+ v_-$ and $u_- := \lim_{t\to+\infty} T_t^- v_+$. Then $u_- \in \mathcal{P}_-$, which implies $u_- \geq \hat{\varphi}_\infty$. By Proposition 2.8 (3), $v_+ \leq v_-$. Then we have $T_t^- v_+ \leq T_t^- v_- = v_-$. Taking $t \to +\infty$ we get $u_- \leq v_-$. Therefore, $v_- \geq u_- \geq \hat{\varphi}_\infty$. □

So far, we complete the proof of Theorem 1.

*4.3. On Example (1.5)*

The Hamiltonian of (1.4) is formulated as

$$H(x,u,p)=\frac{p^2}{2}+\sin x\cdot u+\cos 2x-1. \tag{4.1}$$

We first show $c_0=0$. Assume (1.4) admits a smooth subsolution $u_0$ when $c<0$, then we have $|u_0'(0)|^2\leq 2c<0$, which is impossible. When $c=0$, the constant function $\varphi\equiv 0$ is a subsolution of (1.4). Therefore $c_0=0$. By Proposition 3.5, there is a solution $u_-$ of (1.4) given by

$$u_-:=\lim_{t\to+\infty}T_t^-\varphi.$$

Since $T_t^-\varphi\geq\varphi$, then $u_-\geq 0$.

We then divide the proof into the following steps:

- In Step 1, we discuss the dynamical behavior of the contact Hamiltonian flow $\Phi_t^H$ generated by $H(x,u,p)$, which is restricted on a two dimensional energy shell $M^0$.
  - In Step 1.1, we show that the non-wandering set of $\Phi_t^H$ consists of four fixed points;
  - In Step 1.2, we classify these fixed points by linearization;
  - In Step 1.3, we show that for each solution $v_-$ of (1.4), the $\alpha$-limit set of any $(v_-,L,0)$-calibrated curve $\gamma:(-\infty,0]\to\mathbb{S}^1$ with $\gamma(0)\neq\pi/2$ and $3\pi/2$ can only be 0 or $\pi$. We only focus on the projected $\alpha$-limit set defined on $\mathbb{S}^1$. For simplicity, we define

    $$\alpha(\gamma):=\{x\in\mathbb{S}^1:\ \text{there exists a sequence } t_n\to-\infty \text{ such that } |\gamma(t_n)-x|\to 0\},$$

    where $\gamma:(-\infty,0]\to\mathbb{S}^1$ is a $(v_-,L,0)$-calibrated curve. Moreover, we check the constant curves $\gamma(t)\equiv 0,\pi$ are calibrated curves, which implies $v_-(0)=v_-(\pi)=0$, $v_-'(0)=v_-'(\pi)=0$.
- In Step 2, we prove the uniqueness of the solution $v_-$ of (1.4).
  - In Step 2.1, we prove that $v_-$ is unique near 0 and $\pi$;
  - In Step 2.2, we prove that $v_-$ is unique on $[\pi,2\pi)$ by the comparison along calibrated curves via the Gronwall inequality. The uniqueness of $v_-$ on $[0,\pi]$ is guaranteed by the comparison principle for the Dirichlet problem.

**Step 1. The dynamical behavior of the contact Hamiltonian flow.**

For each solution $v_-$ of (1.4), let $\gamma:(-\infty,0]\to\mathbb{S}^1$ be a $(v_-,L,0)$-calibrated curve. Similar to the analysis at the beginning of [11, Section 3.2], the derivative $v_-'(\gamma(t))$ exists for each $t\in(-\infty,0)$ and the orbit $(\gamma(t),v_-(\gamma(t)),v_-'(\gamma(t)))$ satisfies the contact Hamilton equations generated by the Hamiltonian $H(x,u,p)$ defined in (4.1). Then the proof of the uniqueness of the solution of (1.4) is related to the contact Hamiltonian flow $\Phi_t^H$ generated by $H(x,u,p)$.

Since $c_0=0$ and $H(\gamma(t),v_-(\gamma(t)),v_-'(\gamma(t)))=0$ for $t\in(-\infty,0)$, we discuss the flow on the two dimensional energy shell

$$M^0:=\{(x,u,p)\in T^*\mathbb{S}^1\times\mathbb{R}:\ H(x,u,p)=0\}.$$

Note that along the contact Hamiltonian flow, we have $dH/dt = -H\partial H/\partial u$, which equals to zero on the set $M^0$. Thus, $M^0$ is an invariant set under the action of $\Phi_t^H$. Since we are interested in the orbit $(\gamma(t), v_-(\gamma(t)), v_-'(\gamma(t)))$, we then consider the flow $\Phi_t^H$ restrict on $M^0$. The contact Hamilton equations then reduce to

$$\begin{cases} \dot{x} = p, \\ \dot{p} = -(\cos x \cdot u - 2\sin 2x) - \sin x \cdot p, \\ \dot{u} = p^2. \end{cases} \tag{4.2}$$

**Step 1.1. The non-wandering set.** We first consider the non-wandering set $\Omega$ of $\Phi_t^H|_{M^0}$. Suppose there is an orbit $(x(t), u(t), p(t))$ belongs to $\Omega$. Since $\dot{u} = p^2 \geq 0$, $u(t)$ equals to a constant $c_u$ and $p(t) \equiv 0$. By $\dot{x}(t) = p(t) = 0$, $x(t)$ also equals to a constant $c_x$. By $H(x, u, p) = 0$ and $p = 0$, we have

$$\sin x \cdot u + \cos 2x - 1 = 0.$$

By $p = 0$ and $\dot{p} = 0$ we have

$$\cos x \cdot u - 2\sin 2x = 0.$$

A direct calculation shows that the only non-wandering points are

$$P_1 = (0, 0, 0), \quad P_2 = (\pi, 0, 0), \quad P_3 = (\frac{\pi}{2}, 2, 0), \quad P_4 = (\frac{3\pi}{2}, -2, 0).$$

**Step 1.2. The classification of fixed points.** We then consider the dynamical behavior of $\Phi_t^H|_{M^0}$ near the fixed points. After a translation, we put the fixed points to be the origin. Near the points $P_1$ and $P_2$, the linearized equation of (4.2) is

$$\dot{x} = p, \quad \dot{p} = 4x, \quad \dot{u} = 0.$$

Thus, $P_1$ and $P_2$ are hyperbolic fixed points for the dynamical system $\Phi_t^H|_{M^0}$. Near the points $P_3$ and $P_4$, the linearized equations of (4.2) are

$$\dot{x} = p, \quad \dot{p} = -2x - p, \quad \dot{u} = 0$$

and

$$\dot{x} = p, \quad \dot{p} = -2x + p, \quad \dot{u} = 0$$

respectively. Thus, $P_3$ is a stable focus, and $P_4$ is an unstable focus.

**Step 1.3. The $\alpha$-limit set of calibrated curves.** The $\alpha$-limit set of a $(v_-, L, 0)$-calibrated curve $\gamma$ is contained in the projection of $\Omega$. If $\gamma$ itself is not a fixed point, and the $\alpha$-limit of $\gamma$ is a focus, then there are two constants $t_1 < t_2 < 0$ with $\gamma(t_1) = \gamma(t_2)$ such that $v_-'(\gamma(t_1)) \neq v_-'(\gamma(t_2))$, which is impossible. In other words, the obits near a focus can not form a 1-graph. Thus, the

$\alpha$-limit of $\gamma : (-\infty, 0] \to \mathbb{S}^1$ with $\gamma(0) \neq \pi/2, 3\pi/2$ can only be either 0 or $\pi$. For constant curve $\gamma : (-\infty, 0] \to \mathbb{S}^1$ with $\gamma(t) \equiv x_0$ and $x_0$ equals to either 0 or $\pi$, we have

$$v_-(x_0) - v_-(x_0) = 0 = \int_0^t L(x_0, v_-(x_0), 0)ds,$$

where

$$L(x, u, \dot{x}) = \frac{\dot{x}^2}{2} - \sin x \cdot u - \cos 2x + 1$$

is the Lagrangian corresponding to $H(x, u, p)$. Then the constant curve $\gamma$ is a $(v_-, L, 0)$-calibrated curve. We then have

$$\lim_{t\to-\infty} v_-(\gamma(t)) = v_-(0) = v_-(\pi) = c_u = 0,$$

and

$$\lim_{t\to-\infty} v'_-(\gamma(t)) = v'_-(0) = v'_-(\pi) = 0.$$

**Step 2. The uniqueness of the solution $v_-$ of (1.4).**
**Step 2.1.** For $x \in \mathbb{S}^1 \backslash \{\pi/2, 3\pi/2\}$, let $\gamma : (-\infty, 0] \to \mathbb{S}^1$ with $\gamma(0) = x$ be a $(v_-, L, 0)$-calibrated curve. We claim that there is a constant $\delta > 0$ such that for $x \in [0, \delta]$, the $\alpha$-limit of the calibrated curve $\gamma$ is 0. If not, the $\alpha$-limit of $\gamma$ is $\pi$ for all $x \in (0, \pi]$. Then $v_-$ is decreasing on $(0, \pi]$, since $v_-$ is increasing along $\gamma$ by the last equality of (4.2). By Step 1.3, $v_-(0) = v_-(\pi) = 0$, we get $v_- \equiv 0$ on $[0, \pi]$, which is impossible. By similar arguments, we conclude that there is a constant $\delta > 0$ such that the $\alpha$-limit of $\gamma$ is 0 for $x \in [0, \delta] \cup [2\pi - \delta, 2\pi)$, and the $\alpha$-limit of $\gamma$ is $\pi$ for $x \in [\pi - \delta, \pi + \delta]$. Shrink $\delta$ if necessary, the 1-graph $(x, v_-(x), v'_-(x))$ coincides with the local unstable manifold of $P_1$ (resp. $P_2$) corresponding to the restricted flow $\Phi_t^H|_{M^0}$ when $x \in [0, \delta] \cup [2\pi - \delta, 2\pi)$ (resp. $x \in [\pi - \delta, \pi + \delta]$). Therefore, the solution $v_-$ is unique on $[0, \delta] \cup [2\pi - \delta, 2\pi) \cup [\pi - \delta, \pi + \delta]$.
**Step 2.2.** Since $\sin x \geq \sin \delta > 0$ for $x \in [\delta, \pi - \delta]$, by the uniqueness of the solution of the Dirichlet problem (cf. [3, Theorem 3.3]), $v_-$ is unique on $[0, \pi]$. It remains to consider the uniqueness of $v_-$ for $x \in [\pi, 2\pi)$. Assume that there are two solutions $u_-$ and $v_-$ satisfying $u_-(x) > v_-(x)$ at some point $x \in (\pi + \delta, 3\pi/2)$. Let $\gamma$ be a $(v_-, L, 0)$-calibrated curve with $\gamma(0) = x$. Without any loss of generality, we assume the $\alpha$-limit of $\gamma$ is $\pi$. We take $t_0 < 0$ such that $\gamma(t_0) = \pi + \delta$, and define

$$G(s) := u_-(\gamma(s)) - v_-(\gamma(s)), \quad s \in [t_0, 0].$$

Then $G(t_0) = 0$ and $G(0) > 0$. By continuity, there is $\sigma_0 \in [t_0, 0)$ such that $G(\sigma_0) = 0$ and $G(\sigma) > 0$ for all $\sigma \in (\sigma_0, 0]$. By definition we have

$$u_-(\gamma(\sigma)) - u_-(\gamma(\sigma_0)) \leq \int_{\sigma_0}^{\sigma} L(\gamma(s), u_-(\gamma(s)), \dot{\gamma}(s))ds,$$

and

$$v_-(\gamma(\sigma)) - v_-(\gamma(\sigma_0)) = \int_{\sigma_0}^{\sigma} L(\gamma(s), v_-(\gamma(s)), \dot\gamma(s))ds,$$

which implies

$$G(\sigma) \le \int_{\sigma_0}^{\sigma} G(s)ds.$$

By the Gronwall inequality, we have $G(\sigma) \equiv 0$ for all $\sigma \in (\sigma_0, 0]$, which contradicts $u_-(x) > v_-(x)$. The case $x \in (3\pi/2, 2\pi - \delta)$ is similar. By the continuity of $v_-$ at $3\pi/2$, we finally conclude that the solution is unique on $[\pi, 2\pi)$.

**Remark 4.2.** The method introduced in this section can be generalized to the following case

$$H(x, Du) + \lambda(x)u = c, \quad x \in \mathbb{S}^1,$$

where $\lambda(x)$ and $H(x, p)$ are of class $C^3$ and

(i) the zero points of $\lambda(x)$ are $x_1$ and $x_2$, and $\lambda'(x) \neq 0$ at $x_1$ and $x_2$;
(ii) $H(x, p)$ is strictly convex and superlinear in $p$, $H(x, p) \equiv H(x, -p)$,

$$\max_{x\in\mathbb{S}^1} H(x, 0) = 0$$

and the maximum is achieved at $x_1$ and $x_2$, and the Hessian matrix of $H$ is negative definite at $(x_1, 0)$ and $(x_2, 0) \in T^*\mathbb{S}^1$;
(iii) for all $x \in \mathbb{S}^1$, let $\gamma : (-\infty, 0] \to \mathbb{S}^1$ with $\gamma(0) = x$ be a calibrated curve, then the $\alpha$-limit of $\gamma$ is either $x_1$ or $x_2$.

By (ii), $H(x, p) \ge H(x, 0)$, where the equality holds if and only if $p = 0$. By the argument at the beginning of this section, it is direct to see the critical value $c_0 = 0$. Now let $c = 0$. The contact Hamilton equations for $\Phi_t^H|_{M^0}$ are

$$\begin{cases} \dot x = \dfrac{\partial H}{\partial p}(x, p), \\ \dot p = -\dfrac{\partial H}{\partial x}(x, p) - \lambda'(x)u - \lambda(x)p, \\ \dot u = \dfrac{\partial H}{\partial p}(x, p)p. \end{cases} \tag{4.3}$$

By (ii), $\dot u \ge 0$ and the equality holds if and only if $p = 0$. By the second equation in (4.3), there is only one non-wandering point of $\Phi_t^H|_{M^0}$ over $x_1$ (resp. $x_2$)

$$P_1 = (x_1, 0, 0) \quad (\text{resp. } P_2 = (x_2, 0, 0))$$

Note that

$$L(x,0) = \sup_{p \in T^*\mathbb{S}^1} -H(x,p) = -\inf_{p \in T^*\mathbb{S}^1} H(x,p) = -H(x,0).$$

Similar to Step 1.3 above, we have $v_-(x_1) = v_-(x_2) = 0$ for each solution $v_-$. Near the points $P_1$ and $P_2$, the linearised equation is

$$\dot{x} = \frac{\partial^2 H}{\partial x \partial p} x + \frac{\partial^2 H}{\partial p^2} p, \quad \dot{p} = -\frac{\partial^2 H}{\partial x^2} x - \frac{\partial^2 H}{\partial x \partial p} p, \quad \dot{u} = 0.$$

By (ii), $P_1$ and $P_2$ are hyperbolic fixed points. By (iii) and $\dot{u} \geq 0$, the solution is unique near $x_1$ and $x_2$. The remaining proof is similar to Step 2.2 above, we omit it for brevity.

## 5. Large time behavior of the solution of (CP)

Let us recall $u_{\max}$ (resp. $u^+_{\min}$) be the maximal solution (resp. minimal forward weak KAM solution) of ($E_0$). These two solutions play important roles in characterizing the large time behavior of the solution of (CP).

### 5.1. Above the maximal solution

Let $\varphi \geq u_{\max}$. Then $T^-_t \varphi \geq u_{\max}$. Combining with Lemma 3.2(1), $T^-_t \varphi(x)$ has a bound independent of $t$. Then the pointwise limit

$$\bar{u}(x) := \limsup_{t \to +\infty} T^-_t \varphi(x)$$

exists.

Assume ($\star$) holds. By Proposition 1.6, the family $\{T^-_t \varphi(x)\}_{t \geq 1}$ is equi-Lipschitz in $x$. We denote by $\kappa$ the Lipschitz constant of $T^-_t \varphi(x)$ in $x$. Since

$$|\sup_{s \geq t} T^-_s \varphi(x) - \sup_{s \geq t} T^-_s \varphi(y)| \leq \sup_{s \geq t} |T^-_s \varphi(x) - T^-_s \varphi(y)| \leq \kappa d(x,y),$$

the limiting procedure

$$\bar{u}(x) = \lim_{t \to +\infty} \sup_{s \geq t} T^-_s \varphi(x)$$

is uniform in $x$. Thus, the function $\bar{u}(x)$ is Lipschitz continuous. We assert that $\bar{u}$ is a subsolution. If the assertion is true, by Proposition 3.5, $\lim_{t \to +\infty} T^-_t \bar{u}(x)$ exists, and it is a solution. Since $T^-_t \varphi \geq u_{\max}$, we have $\bar{u} \geq u_{\max}$. Thus, $\lim_{t \to +\infty} T^-_t \bar{u} = u_{\max}$. Based on Section 4.1, the lower half limit $\check{\varphi} = u_{\max}$. By the definition of $\check{\varphi}$, we have

$$\liminf_{t \to +\infty} T^-_t \varphi(x) \geq \check{\varphi}(x) = u_{\max}.$$

On the other hand,

$$\limsup_{t\to+\infty} T_t^-\varphi(x) = \bar{u}(x) \leq \lim_{t\to+\infty} T_t^-\bar{u}(x) = u_{\max}(x).$$

It follows that $\lim_{t\to+\infty} T_t^-\varphi = u_{\max}$ uniformly on $M$.

It remains to prove $\bar{u}$ is a subsolution. By Proposition 2.5, we only need to show $T_t^-\bar{u}$ is increasing in $t$.

We claim that for every $\varepsilon > 0$, there exists a constant $s_0 > 0$ independent of $x$ such that for any $s \geq s_0$,

$$T_s^-\varphi(x) \leq \bar{u}(x) + \varepsilon.$$

Fixing $x \in M$, by definition of lim sup, for every $\varepsilon > 0$, there is $s_0(x) > 0$ such that for any $s \geq s_0(x)$,

$$T_s^-\varphi(x) \leq \bar{u}(x) + \frac{\varepsilon}{3}.$$

Taking $r := \frac{\varepsilon}{3\kappa}$. For $s \geq s_0(x)$, we have

$$\begin{aligned} T_s^-\varphi(y) &\leq T_s^-\varphi(x) + \kappa d(x,y) \leq \bar{u}(x) + \frac{\varepsilon}{3} + \kappa d(x,y) \\ &\leq \bar{u}(y) + \frac{\varepsilon}{3} + 2\kappa d(x,y) \leq \bar{u}(y) + \varepsilon, \quad \forall y \in B_r(x). \end{aligned}$$

Since $M$ is compact, there are finite points $x_i \in M$ such that for each $y \in M$, there is a point $x_i$ such that $y \in B_r(x_i)$. Let $s_0 := \max_i s_0(x_i)$ and the claim is proved.

By Proposition 2.2, for each $t > 0$ we have

$$T_t^-(T_s^-\varphi(x)) \leq T_t^-(\bar{u}(x) + \varepsilon) \leq T_t^-\bar{u}(x) + \varepsilon e^{\lambda_0 t},$$

where $\lambda_0 := \|\lambda(x)\|_\infty > 0$. Taking the limit $s \to +\infty$, we have

$$\bar{u}(x) = \limsup_{s\to+\infty} T_t^-(T_s^-\varphi(x)) \leq T_t^-\bar{u}(x) + \varepsilon e^{\lambda_0 t}.$$

Letting $\varepsilon \to 0+$, we get $\bar{u}(x) \leq T_t^-\bar{u}(x)$, which means $T_t^-\bar{u}(x)$ is increasing in $t$.

### 5.2. *Below the minimal solution*

We have proved that for each $\varphi \geq u_{\max}$, $\lim_{t\to+\infty} T_t^-\varphi = u_{\max}$ uniformly on $M$. Combining with Proposition 2.4 and Proposition 2.7, one has

**Lemma 5.1.** *Let $\varphi \in C(M)$. If $\varphi \leq u_{\min}^+$, then $\lim_{t\to+\infty} T_t^+\varphi = u_{\min}^+$ uniformly on $M$.*

**Lemma 5.2.** *Let $\varphi \in C(M)$ and there is a point $x_0 \in M$ such that $\varphi(x_0) < u_{\min}^+(x_0)$, then $T_t^-\varphi(x)$ tends to $-\infty$ uniformly on $M$ as $t \to +\infty$.*

**Proof.** We first prove that $\min_{x\in M} T_t^-\varphi(x)$ tends to $-\infty$ as $t\to+\infty$. We argue by contradiction. Assume there is a constant $K_1$ and a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $T_{t_n}^-\varphi\geq K_1$. By Lemma 3.2, $T_{t_n}^-\varphi$ also has a upper bound independent of $t$. Thus, the function $v_n(x):=T_{t_n}^-\varphi(x)$ is bounded continuous for each $n$. By Proposition 2.6, we have $\varphi(x_0)\geq T_{t_n}^+v_n(x_0)$. By Proposition 3.4, all subsolutions are uniformly bounded. Denote by $K_2$ their lower bound. Let $K':=\min\{K_1,K_2\}$, then $T_{t_n}^+v_n\geq T_{t_n}^+K'$. By Lemma 3.2(2), $T_t^+K'$ has a lower bound independent of $t$. Since $K'\leq K_2$, $T_t^+K'$ is smaller than every forward weak KAM solution of ($E_0$). By Lemma 5.1, $\lim_{t\to+\infty}T_t^+K'$ exists and it equals to $u^+_{\min}$. We conclude

$$u^+_{\min}(x_0)\leq \limsup_{t_n\to+\infty} T_{t_n}^+v_n(x_0)\leq\varphi(x_0)<u^+_{\min}(x_0),$$

which leads to a contradiction.

We then prove that $T_t^-\varphi(x)$ tends to $-\infty$ uniformly as $t\to+\infty$. Let $W(x)$ be the inverse function of $x\mapsto xe^x$. Taking $0<\eta\leq W(1)/\lambda_0$. We define $K(t):=\min_{x\in M}T_t^-\varphi(x)$, which tends to $-\infty$ as $t\to+\infty$. We take an arbitrary $x\in M$. If $T_{t+\eta}^-\varphi(x)\leq K(t)$, then the proof is finished. So we assume $T_{t+\eta}^-\varphi(x)>K(t)$. Let $x_t$ be the minimal point of $T_t^-\varphi$. Taking a geodesic $\alpha:[0,\eta]\to M$ with $\alpha(0)=x_t$, $\alpha(\eta)=x$ and constant speed $\|\dot\alpha\|\leq \mathrm{diam}(M)/\eta$. By continuity, there is $\sigma\in[0,\eta)$ such that $T_{t+\sigma}^-\varphi(\alpha(\sigma))=K(t)$ and $T_{t+s}^-\varphi(\alpha(s))>K(t)$ for all $s\in(\sigma,\eta]$. Then

$$\begin{aligned}
T_{t+s}^-\varphi(\alpha(s))&\leq T_{t+\sigma}^-\varphi(\alpha(\sigma))+\int_\sigma^s\Big[L(\alpha(\tau),\dot\alpha(\tau))-\lambda(\alpha(\tau))\cdot T_{t+\tau}^-\varphi(\alpha(\tau))+c\Big]d\tau\\
&\leq K(t)+\int_\sigma^s\Big[L(\alpha(\tau),\dot\alpha(\tau))-\lambda_0K(t)+c\Big]d\tau+\lambda_0\int_\sigma^s\Big[T_{t+\tau}^-\varphi(\alpha(\tau))-K(t)\Big]d\tau\\
&\leq K(t)+\bar C_L\eta-\lambda_0\eta K(t)+\lambda_0\int_\sigma^s\Big[T_{t+\tau}^-\varphi(\alpha(\tau))-K(t)\Big]d\tau,
\end{aligned}$$

where

$$\bar C_L:=\max_{x\in M,\|\dot x\|\leq\mathrm{diam}(M)/\eta}|L(x,\dot x)+c|$$

is finite for a fixed $\eta$ by the assumption ($\star$). By the Gronwall inequality, we have

$$T_{t+s}^-\varphi(\alpha(s))\leq \bar C_L\eta e^{\lambda_0\eta}+(1-\lambda_0\eta e^{\lambda_0\eta})K(t).$$

Since $\eta\leq W(1)/\lambda_0$, we have $1-\lambda_0\eta e^{\lambda_0\eta}>0$. Taking $s=\eta$, we finally conclude that $T_t^-\varphi(x)$ tends to $-\infty$ as $t\to+\infty$. □

So far, we complete the proof of Theorem 2.

### 5.3. *Proof of Theorem 3*

According to Proposition A.6, for $c \geq c_0$, ($E_0$) has a Lipschitz subsolution. Let $u_0$ be a subsolution of ($E_0$) with $c = c_0$. For $c > c_0$, there holds

$$T_t^+ u_0 < u_0 < T_t^- u_0.$$

One can construct two different solutions $u_-$ and $v_-$ of ($E_0$) from $u_0$ by Proposition A.7. Precisely, we have

$$u_- = \lim_{t \to +\infty} T_t^- u_0, \; u_+ = \lim_{t \to +\infty} T_t^+ u_0, \;\; v_- = \lim_{t \to +\infty} T_t^- u_+. \tag{5.1}$$

It follows that $u_+ < u_0 < u_-$.

**Lemma 5.3.** *Let $c > c_0$. For each $\alpha \in (0, 1]$ and each solution $w_-$ of ($E_0$), the convex combination*

$$u_\alpha := \alpha u_0 + (1 - \alpha) w_-$$

*is a strict subsolution of ($E_0$). In particular, we have $T_t^+ u_\alpha < u_\alpha < T_t^- u_\alpha$.*

**Proof.** Since $u_0$ is a Lipschitz subsolution of ($E_0$) with $c = c_0$, we have

$$H(x, Du_0(x)) + \lambda(x) u_0(x) + (c - c_0) \leq c, \quad a.e \; x \in M.$$

Since $w_-$ is a solution of ($E_0$), we have

$$H(x, Dw_-(x)) + \lambda(x) w_-(x) = c, \quad a.e. \; x \in M.$$

Therefore

$$\begin{aligned} &\alpha H(x, Du_0(x)) + (1 - \alpha) H(x, Dw_-(x)) \\ &\qquad + \lambda(x)\Big(\alpha u_0(x) + (1 - \alpha) w_-(x)\Big) + \alpha(c - c_0) \leq c, \quad a.e. \; x \in M. \end{aligned}$$

By the convexity of $H(x, p)$ with respect to $p$, the Jensen's inequality gives

$$H(x, Du_\alpha(x)) + \lambda(x) u_\alpha(x) \leq (1 - \alpha) c + \alpha c_0, \quad a.e. \; x \in M.$$

Let $\epsilon_0 := \alpha(c - c_0) > 0$. Then

$$H(x, Du_\alpha(x)) + \lambda(x) u_\alpha(x) + \epsilon_0 \leq c, \quad a.e. \; x \in M.$$

By Lemma A.2, $T_t^+ u_\alpha < u_\alpha < T_t^- u_\alpha$. □

**Lemma 5.4.** *Let $c > c_0$. Define $u_-$ and $v_-$ as in (5.1). Then $u_-$ is the maximal solution of ($E_0$), and $v_-$ is the minimal solution of ($E_0$).*

**Proof.** In the first step, we prove that there is no solution $w_-$ different from $u_-$ such that $w_- \geq u_-$. Assume that there is such a solution $w_-$. Since $u_0 < u_- \leq w_-$, there is $\alpha \in (0, 1)$ such that $u_\alpha = \alpha u_0 + (1-\alpha)w_-$ satisfies

$$\min_{x\in M}(u_-(x) - u_\alpha(x)) = 0.$$

Let $x_0 \in M$ be the point at which the above minimum is attained. Then

$$T_t^- u_\alpha \leq T_t^- u_-.$$

By Lemma 5.3, we have $T_t^- u_\alpha(x_0) > u_\alpha(x_0) = u_-(x_0) = T_t^- u_-(x_0)$, which leads to a contradiction.

We then turn to prove that $u_-$ is the maximal solution, that is, $w_- \leq u_-$ for all solutions $w_-$. Assume that there is a solution $w_-$ such that

$$\max_{x\in M}(w_-(x) - u_-(x)) > 0.$$

Let $y_0 \in M$ be the point at which the above maximum is attained. Then the function $\bar{u}(x) := \max\{u_-(x), w_-(x)\}$ is a subsolution. By Proposition 3.5, we get a solution

$$\bar{w}_- := \lim_{t\to+\infty} T_t^- \bar{u} \geq \bar{u} \geq u_-.$$

We also have

$$\bar{w}_-(y_0) \geq \bar{u}(y_0) = w_-(y_0) > u_-(y_0).$$

Then $\bar{w}_-$ is different from $u_-$ and $\bar{w}_- \geq u_-$. This contradicts what we got in the first step.

Similar to the argument above, we conclude that $u_+$ is the minimal forward weak KAM solution of ($E_0$). By Lemma 4.1, $v_-$ is the minimal solution of ($E_0$). $\square$

Let us recall $u_0$ is a subsolution of ($E_0$) with $c = c_0$. For $c > c_0$, there holds

$$T_t^+ u_0 < u_0 < T_t^- u_0.$$

By Proposition 2.2(1) and Proposition 2.6, we have

$$T_{t+s}^- u_0 \geq T_{t+s}^- \circ T_t^+ u_0 = T_s^- \circ (T_t^- \circ T_t^+ u_0) \geq T_s^- u_0$$

for all $t, s \geq 0$. Letting $s \to +\infty$, we have

$$\lim_{s\to+\infty} T_{t+s}^- \circ T_t^+ u_0 = u_{\max}, \tag{5.2}$$

for each $t > 0$. Let $\varphi \in C(M)$ satisfy $u_{\min}^+ < \varphi \leq u_{\max}$. Since $u_{\min}^+ = \lim_{t\to+\infty} T_t^+ u_0$ by Lemma 5.4, there is $t_0 > 0$ such that $T_{t_0}^+ u_0 \leq \varphi$ on $M$. Then we have

$$T^-_{t_0+s} \circ T^+_{t_0} u_0 \leq T^-_{t_0+s} \varphi \leq u_{\max}.$$

Letting $s \to +\infty$ and by (5.2), we have

$$\lim_{t \to +\infty} T^-_t \varphi = u_{\max}.$$

Now we assume ($\star$) holds. Then for each $\varphi > u^+_{\min}$, there is $\varphi_1$ and $\varphi_2$ such that

$$\varphi_1 \geq u_{\max}, \quad u^+_{\min} < \varphi_2 \leq u_{\max}, \quad \varphi_2 \leq \varphi \leq \varphi_1.$$

Then we have $T^-_t \varphi_2 \leq T^-_t \varphi \leq T^-_t \varphi_1$. Since $\lim_{t \to +\infty} T^-_t \varphi_i = u_{\max}$ for $i = 1, 2$, we have

$$\lim_{t \to +\infty} T^-_t \varphi = u_{\max}.$$

The proof of Theorem 3 is now complete.

## Data availability

No data was used for the research described in the article.

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## Appendix A. Auxiliary results

### A.1. *Proof of Proposition 2.5*

**Lemma A.1.** *If $\varphi$ is a Lipschitz subsolution of ($B_0$), then $\varphi \prec L$.*

**Proof.** Without loss of generality, we assume $M$ is an open set of $\mathbb{R}^n$. In fact, for each absolutely continuous curve $\gamma : [0, t] \to M$, we cover it by local coordinate charts. Clearly, there exists $N \in \mathbb{N}$ such that $[0, t] = \cup_{i=0}^{N-1} [t_i, t_{i+1}]$ with $t_0 = 0$, $t_N = t$, such that $\gamma|_{[t_i, t_{i+1}]}$ is contained in an open subset of $\mathbb{R}^n$.

By [9, Proposition 2.4], there is a function $q \in L^\infty([0, t], \mathbb{R}^n)$ such that for almost all $s \in [0, t]$, we have

$$\frac{d}{ds} \varphi(\gamma(s)) = q(s) \cdot \dot{\gamma}(s),$$

and the vector $q(s)$ belongs to $\partial_c \varphi(\gamma(s))$. Here we recall the definition of the Clarke's generalized gradient

$$\partial_c \varphi(x) := \bigcap_{r>0} \overline{\text{co}} \{D\varphi(y) : \ y \in B(x, r), \text{ and } \varphi \text{ is differentiable at } y\},$$

where $\overline{\text{co}}$ stands for the closure of the convex combination. Since $\varphi$ is a Lipschitz subsolution of ($B_0$), if $\varphi$ is differentiable at $y$, we have

$$H(y, \varphi(y), D\varphi(y)) \leq 0.$$

By the convexity of $H$ with respect to $p$, and the definition of $\partial_c\varphi(x)$, we have

$$H(x, \varphi(x), q) \leq 0, \quad \forall q \in \partial_c\varphi(x).$$

We conclude that

$$\begin{aligned}\varphi(\gamma(t)) - \varphi(\gamma(0)) &= \int_0^t \frac{d}{ds}\varphi(\gamma(s))ds = \int_0^t q(s)\cdot\dot{\gamma}(s)ds \\ &\leq \int_0^t \Big[L(\gamma(s), \varphi(\gamma(s)), \dot{\gamma}(s)) + H(\gamma(s), \varphi(\gamma(s)), q(s))\Big]ds \\ &\leq \int_0^t L(\gamma(s), \varphi(\gamma(s)), \dot{\gamma}(s))ds,\end{aligned}$$

which implies $\varphi \prec L$. □

**Lemma A.2.** *If $\varphi \prec L$, then for each $t \geq 0$, we have $T_t^-\varphi \geq \varphi \geq T_t^+\varphi$. Moreover, if there exists $\epsilon_0 > 0$ such that for a.e. $x \in M$,*

$$H(x, u, Du) + \epsilon_0 \leq 0,$$

*then*

$$T_t^+\varphi < \varphi < T_t^-\varphi.$$

**Proof.** In the following, we only prove $T_t^-\varphi \geq \varphi$ for each $t \geq 0$, since the proof of $T_t^+\varphi \leq \varphi$ is similar. By contradiction, we assume there exists $x_0 \in M$ such that $\varphi(x_0) > T_t^-\varphi(x_0)$. Let $\gamma : [0, t] \to M$ be a minimizer of $T_t^-\varphi$ with $\gamma(t) = x_0$, i.e.

$$T_t^-\varphi(x) = \varphi(\gamma(0)) + \int_0^t L(\gamma(\tau), T_\tau^-\varphi(\gamma(\tau)), \dot{\gamma}(\tau))d\tau. \tag{A.1}$$

Let $F(\tau) := \varphi(\gamma(\tau)) - T_\tau^-\varphi(\gamma(\tau))$. Since $F(t) > 0$ and $F(0) = 0$, then one can find $s_0 \in [0, t)$ such that $F(s_0) = 0$ and $F(s) > 0$ for $s \in (s_0, t]$. A direct calculation shows

$$F(s) \leq \Theta \int_{s_0}^s F(\tau)d\tau,$$

which implies $F(s) \leq 0$ for $s \in (s_0, t]$ from the Gronwall inequality. It contradicts $F(t) > 0$.

Next, we assume there exists $\epsilon_0 > 0$ such that for a.e. $x \in M$,

$$H(x, u, Du) + \epsilon_0 \leq 0.$$

Let us denote

$$\tilde{L}(x, u, \dot{x}) := L(x, u, \dot{x}) - \epsilon_0,$$

and let $\tilde{T}_t^-$ be the Lax-Oleinik semigroup associated to $\tilde{L}$. By a similar argument above, we have $\tilde{T}_t^- \varphi \geq \varphi$ and $\tilde{T}_t^+ \varphi \leq \varphi$. Note that $\tilde{L} < L$. Using a similar argument as [17, Proposition 3.1], $\tilde{T}_t^- \varphi < T_t^- \varphi$ and $\tilde{T}_t^+ \varphi > T_t^+ \varphi$ for each $t > 0$. Therefore, $T_t^- \varphi > \varphi$ and $T_t^+ \varphi < \varphi$ for each $t > 0$. This completes the proof. □

**Lemma A.3.** *If for each $t > 0$, $T_t^- \varphi \geq \varphi$, then $\varphi$ is a Lipschitz subsolution of ($E_0$).*

**Proof.** Fix $T > 0$, by assumption we have $T_t^- \varphi \geq \varphi$ for each $t \in [0, T]$. By [14], there is a constant $R_0 > 0$ depending on $T$ and $\|D\varphi\|_\infty$, such that $\|DT_t^- \varphi(x)\|_\infty \leq R_0$. Let $R := \max\{R_0, \|D\varphi\|_\infty\}$, we make a modification

$$H_R(x, u, p) := H(x, u, p) + \max\{\|p\|^2 - R^2, 0\}.$$

Then $T_t^- \varphi$ is also the solution of ($A_0$) with $H$ replaced by $H_R$. One can prove that the Lagrangian $L_R$ corresponding to $H_R$ is continuous. By the uniqueness of the solution of ($A_0$), we have $T_t^- \varphi = T_t^R \varphi$, where $T_t^R \varphi$ is defined by (2.1) with $L$ replaced by $L_R$.

Let $\varphi$ be differentiable at $x \in M$. For each $v \in T_x M$, there is a $C^1$ curve $\gamma : [0, T] \to M$ with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. By assumption for each $t \in [0, T]$, we have

$$\varphi(\gamma(t)) \leq T_t^- \varphi(\gamma(t)) = T_t^R \varphi(\gamma(t)) \leq \varphi(x) + \int_0^t L_R(\gamma(s), T_s^R \varphi(\gamma(s)), \dot{\gamma}(s)) ds.$$

Dividing by $t$ and let $t$ tend to zero, using the continuity of $\gamma$, $L_R$ and $T_t^R \varphi(x)$. We get

$$D\varphi(x) \cdot v \leq L_R(x, \varphi(x), v).$$

Since $v$ is arbitrary, we have

$$H_R(x, \varphi(x), D\varphi(x)) = \sup_{v \in T_x M} \left[ D\varphi(x) \cdot v - L_R(x, \varphi(x), v) \right] \leq 0.$$

Therefore, $\varphi$ is a Lipschitz subsolution of

$$H_R(x, u(x), Du(x)) = 0.$$

By the definition of $H_R$, $\varphi$ is also a Lipschitz subsolution of ($B_0$). □

### A.2. *Proof of Proposition 2.6*

We only prove $\varphi \le T_t^- \circ T_t^+ \varphi$, the other side is similar. We argue by a contradiction. Assume that there is $x \in M$ and $t > 0$ such that

$$T_t^- \circ T_t^+ \varphi(x) < \varphi(x).$$

Let $\gamma : [0,t] \to M$ with $\gamma(t) = x$ be a minimizer of $T_t^- \circ T_t^+ \varphi(x)$, and define

$$F(s) := T_{t-s}^+ \varphi(\gamma(s)) - T_s^- \circ T_t^+ \varphi(\gamma(s)).$$

Then $F(0) = 0$ and $F(t) > 0$. By continuity, there is $\sigma \in [0,t)$ such that $F(\sigma) = 0$ and $F(\tau) > 0$ for all $\tau \in (\sigma, t]$. By definition, for $s \in (\sigma, t]$ we have

$$\begin{aligned} T_s^- \circ T_t^+ \varphi(\gamma(s)) &= T_\sigma^- \circ T_t^+ \varphi(\gamma(\sigma)) + \int_\sigma^s L(\gamma(\tau), T_\tau^- \circ T_t^+ \varphi(\gamma(\tau)), \dot\gamma(\tau)) d\tau \\ &= T_{t-\sigma}^+ \varphi(\gamma(\sigma)) + \int_\sigma^s L(\gamma(\tau), T_\tau^- \circ T_t^+ \varphi(\gamma(\tau)), \dot\gamma(\tau)) d\tau \\ &\ge T_{t-s}^+ \varphi(\gamma(s)) - \int_\sigma^s L(\gamma(\tau), T_{t-\tau}^+ \varphi(\gamma(\tau)), \dot\gamma(\tau)) d\tau \\ &\quad + \int_\sigma^s L(\gamma(\tau), T_\tau^- \circ T_t^+ \varphi(\gamma(\tau)), \dot\gamma(\tau)) d\tau \\ &\ge T_{t-s}^+ \varphi(\gamma(s)) - \Theta \int_\sigma^s F(\tau) d\tau, \end{aligned}$$

which implies

$$F(s) \le \Theta \int_\sigma^s F(\tau) d\tau.$$

By the Gronwall inequality, we have $F(s) \equiv 0$ for $s \in [\sigma, t]$, which contradicts $F(t) > 0$.

### A.3. *Proof of Proposition 1.2*

#### A.3.1. *$c_0$ and subsolutions*

Inspired by [4], we denote

$$c_0 := \inf_{u \in C^\infty(M)} \sup_{x \in M} \Big\{ H(x, Du) + \lambda(x) u \Big\}.$$

**Proposition A.4.** *$c_0$ is finite.*

**Proof.** Choose $u(x) \equiv 0$, then by definition,

$$c_0 \leq \sup_{x \in M} H(x, 0) < +\infty.$$

Let us recall

$$\mathbf{e}_0 := \min_{(x,p) \in T^*M} H(x, p) > -\infty.$$

By the assumption (±), there exists $x_0 \in M$ such that $\lambda(x_0) = 0$. Thus for each $u \in C^\infty(M)$,

$$\begin{aligned} c_0 &= \inf_{u \in C^\infty(M)} \sup_{x \in M} \Big\{ H(x, Du(x)) + \lambda(x)u(x) \Big\} \\ &\geqslant \inf_{u \in C^\infty(M)} \Big\{ H(x_0, Du(x_0)) + \lambda(x_0)u(x_0) \Big\} \\ &= \inf_{u \in C^\infty(M)} H(x_0, Du(x_0)) \geqslant \mathbf{e}_0. \end{aligned}$$

This means $c_0$ is finite. □

**Proposition A.5.** *For $c < c_0$, ($E_0$) has no continuous subsolutions.*

**Proof.** By contradiction, we assume for $c < c_0$, ($E_0$) admits a continuous subsolution $u : M \to \mathbb{R}$. By the definition of the subsolution, for any $p \in D^+u(x)$,

$$H(x, p) \leq c - \lambda(x)u(x) \leq c + \lambda_0 \|u\|_\infty.$$

Combining (CER), one can conclude that $u$ is Lipschitz continuous (see [8, Proposition 1.14] for more details). By [6, Lemma 2.2], for all $\varepsilon > 0$, there exists $u_\varepsilon \in C^\infty(M)$ such that $\|u - u_\varepsilon\|_\infty < \varepsilon$ and for all $x \in M$,

$$H(x, Du_\varepsilon(x)) + \lambda(x)u(x) \leq c + \varepsilon.$$

We choose $\varepsilon = \frac{1}{2(1+\lambda_0)}(c_0 - c) > 0$, then

$$\begin{aligned} & H(x, Du_\varepsilon(x)) + \lambda(x)u_\varepsilon(x) \\ \leqslant\ & H(x, Du_\varepsilon(x)) + \lambda(x)u(x) + \lambda_0 \|u - u_\varepsilon\|_\infty \\ \leqslant\ & c + (1 + \lambda_0)\varepsilon < c_0, \end{aligned}$$

this contradicts the definition of $c_0$. □

*A.3.2. Existence of subsolutions and solutions*

Let us recall that $T_t^{\pm}$ denote the Lax-Oleinik semigroups associated to

$$L(x, \dot{x}) - \lambda(x)u(x) + c.$$

**Proposition A.6.** *For $c \geq c_0$, ($E_0$) has a Lipschitz subsolution. Let $u_0$ be a subsolution of ($E_0$) with $c = c_0$. For $c > c_0$, there holds*

$$T_t^{+}u_0 < u_0 < T_t^{-}u_0.$$

**Proof.** By the definition of $c_0$, there exists $u_n \in C^{\infty}(M)$ such that for all $x \in M$,

$$H(x, Du_n(x)) + \lambda(x)u_n(x) \leqslant c_0 + \frac{1}{n}. \tag{A.2}$$

Namely, $u_n$ is a subsolution of

$$H(x, Du) + \lambda(x)u = c_0 + 1.$$

By Proposition 3.4, $\{u_n\}_{n\geq 1}$ is equi-bounded and equi-Lipschitz continuous. Then by the Ascoli-Arzelà theorem, it contains a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ uniformly converging on $M$ to some $u_0 \in \mathrm{Lip}(M)$. By the stability of subsolutions (see [2, Theorem 5.2.5]), $u_0$ is a subsolution of

$$H(x, Du) + \lambda(x)u = c_0.$$

Moreover, for $c > c_0$ and a.e. $x \in M$, we have

$$H(x, Du_0) + \lambda(x)u_0 + (c - c_0) \leq c.$$

By Lemma A.2,

$$T_t^{+}u_0 < u_0 < T_t^{-}u_0.$$

This completes the proof. □

Combining Propositions A.5, A.6 and 3.5, we conclude that ($E_0$) has a solution if and only if $c \geq c_0$. It remains to prove the following result.

**Proposition A.7.** *($E_0$) has at least two solutions for $c > c_0$.*

**Proof.** By Proposition A.6, if $c > c_0$, there exists a strict Lipschitz subsolution $u_0$ of ($E_0$). Based on Proposition 2.5, for $t > 0$,

$$T_t^{-}u_0(x) > u_0(x), \quad T_t^{+}u_0(x) < u_0(x). \tag{A.3}$$

Denote

$$u_- := \lim_{t\to+\infty} T_t^- u_0(x), \quad u_+ := \lim_{t\to+\infty} T_t^+ u_0(x), \tag{A.4}$$

and

$$v_- := \lim_{t\to+\infty} T_t^- u_+(x). \tag{A.5}$$

By Proposition 3.5, $u_-$ and $v_-$ are solutions of ($E_0$).

It remains to verify $u_- \neq v_-$. By contradiction, we assume $u_- \equiv v_-$ on $M$. In view of (A.5), we have

$$u_- = \lim_{t\to+\infty} T_t^- u_+(x). \tag{A.6}$$

Based on (A.6), it follows from Proposition 2.9 that

$$\mathcal{I}_{u_+} := \{x \in M : \ u_-(x) = u_+(x)\} \neq \emptyset. \tag{A.7}$$

On the other hand, from (A.3) and (A.4), it follows that for any $x \in M$,

$$u_+(x) < u_0(x) < u_-(x), \tag{A.8}$$

which implies

$$\mathcal{I}_{u_+} = \emptyset.$$

This contradicts (A.7). □

*A.4. Proof of Proposition 1.6*

Assume that $H(x,p)$ is continuous and satisfies the condition ($\star$). Then the associated Lagrangian $L(x,\dot{x})$ satisfies

**(CL):** $L(x,\dot{x})$ and $\frac{\partial L}{\partial \dot{x}}(x,\dot{x})$ are continuous;
**(CON):** $L(x,\dot{x})$ is convex in $\dot{x}$, for any $x \in M$;
**(SL):** there is a superlinear function $\eta(r)$ such that $L(x,\dot{x}) \geq \eta(\|\dot{x}\|)$.

With a slight modification, [1, Theorem 2.2] implies

**Lemma A.8.** *(Erdmann condition). For each $(x,t) \in M \times (0,+\infty)$, let $\gamma : [0,t] \to M$ be a minimizer of $T_t^-\varphi(x)$. Set $u_1(s) := T_s^-\varphi(\gamma(s))$ with $s \in [0,t]$, and*

$$E_0(s) := \frac{\partial L}{\partial \dot{x}}(\gamma(s),\dot{\gamma}(s)) \cdot \dot{\gamma}(s) - L(\gamma(s),\dot{\gamma}(s)),$$

*then*

$$E(s) := e^{\int_0^s \lambda(\gamma(r))dr}[E_0(s) + \lambda(\gamma(s))u_1(s)]$$

*satisfies* $\dot{E}(s) = 0$ *a.e on* $[0, t]$.

Based on Lemma A.8, we have

**Theorem A.9.** *The function* $(x, t) \mapsto T_t^- \varphi(x)$ *is locally Lipschitz on* $M \times (0, +\infty)$. *More precisely, given two positive constants* $\delta$ *and* $T$ *with* $\delta < T$. *For each* $\varphi \in C(M)$ *and* $t \in [\delta, T]$, *the Lipschitz constant of* $T_t^- \varphi(x)$ *depends only on* $\|\varphi\|_\infty$, $\delta$ *and* $T$.

**Proof. Step 1. Lipschitz estimate of minimizers.** Given $(x, t) \in M \times [\delta, T]$. In the following, we denote by $\gamma : [0, t] \to M$ a minimizer of $T_t^- \varphi(x)$. We focus on the Lipschitz regularity of the curve $\gamma$. Note that $T_t^-(-\|\varphi\|_\infty) \le T_t^- \varphi \le T_t^- \|\varphi\|_\infty$, $T_t^- \varphi$ is bounded by a constant $K$ depending only on $\|\varphi\|_\infty$ and $T$. We then have

$$
\begin{aligned}
K \ge T_t^- \varphi(x) &= \varphi(\gamma(0)) + \int_0^t \Big[ L(\gamma(s), \dot{\gamma}(s)) - \lambda(\gamma(s)) T_s^- \varphi(\gamma(s)) \Big] ds \\
&\ge -\|\varphi\|_\infty - \lambda_0 K T + \int_0^t L(\gamma(s), \dot{\gamma}(s)) ds.
\end{aligned}
$$

By (SL), there is a constant $D$ such that $L(\gamma(s), \dot{\gamma}(s)) \ge \|\dot{\gamma}(s)\| + D$, then we have

$$
K + (\lambda_0 K + |D|)T + \|\varphi\|_\infty \ge \int_0^t \|\dot{\gamma}(s)\| ds.
$$

Thus, there is $s_0 \in [0, t]$ such that $\|\dot{\gamma}(s_0)\|$ is bounded by a constant depending only on $\|\varphi\|_\infty$, $\delta$ and $T$. Recall

$$
E(s) := e^{\int_0^t \lambda(\gamma(r)) dr} [E_0(s) + \lambda(\gamma(s)) u_1(s)].
$$

By Lemma A.8, $\dot{E}(s) = 0$ a.e. on $[0, t]$. It follows that

$$
E_0(s) \le e^{\lambda T} (|E_0(s_0)| + \lambda_0 K) + \lambda_0 K := F_1.
$$

By (CON) we have

$$
\begin{aligned}
L(\gamma(s), \frac{\dot{\gamma}(s)}{1 + \|\dot{\gamma}(s)\|}) - L(\gamma(s), \dot{\gamma}(s)) &\ge (\frac{1}{1 + \|\dot{\gamma}(s)\|} - 1) \frac{\partial L}{\partial \dot{x}} (\gamma(s), \dot{\gamma}(s)) \cdot \dot{\gamma}(s) \\
&\ge (\frac{1}{1 + \|\dot{\gamma}(s)\|} - 1)(F_1 + L(\gamma(s), \dot{\gamma}(s))).
\end{aligned}
$$

We denote by $K_3$ the bound of $L(x, \dot{x})$ for $\|\dot{x}\| \le 1$. Then we have

$$
L(\gamma(s), \dot{\gamma}(s)) \le 2K_3 + F_1.
$$

By (SL), $\|\dot{\gamma}(s)\|$ is bounded by a constant depending only on $\|\varphi\|_\infty$, $\delta$ and $T$.

**Step 2. Lipschitz estimate of $(x,t) \mapsto T_t^-\varphi(x)$.** We first show that $u(x,t) := T_t^-\varphi(x)$ is locally Lipschitz in $x$. For any $r > 0$ with $2r < \delta$, given $(x_0, t) \in M \times [\delta, T]$ and $x, x' \in B(x_0, r)$, denote by $d_0 := d(x, x') \le 2r < \delta$ the Riemannian distance between $x$ and $x'$, we have

$$u(x',t) - u(x,t) \le \int_{t-d_0}^{t} \Big[L(\alpha(s),\dot{\alpha}(s)) - \lambda(\alpha(s))u(\alpha(s),s)\Big]ds - \int_{t-d_0}^{t} \Big[L(\gamma(s),\dot{\gamma}(s)) - \lambda(\gamma(s))u(\gamma(s),s)\Big]ds,$$

where $\gamma(s)$ is a minimizer of $u(x,t)$ and $\alpha : [t-d_0, t] \to M$ is a geodesic satisfying $\alpha(t-d_0) = \gamma(t-d_0)$ and $\alpha(t) = x'$ with constant speed. By Step 1, the bound of $\|\dot{\gamma}(s)\|$ depends only on $\|\varphi\|_\infty$, $\delta$ and $T$. Since

$$\|\dot{\alpha}(s)\| \le \frac{d(\gamma(t-d_0), x')}{d_0} \le \frac{d(\gamma(t-d_0), x)}{d_0} + 1,$$

and $d(\gamma(t-d_0), x) \le \int_{t-d_0}^{t} \|\dot{\gamma}(s)\| ds$, the bound of $\|\dot{\alpha}(s)\|$ also depends only on $\|\varphi\|_\infty$, $\delta$ and $T$. Exchanging the role of $(x,t)$ and $(x',t)$, one obtains that $|u(x,t) - u(x',t)| \le J_1 d(x,x')$, where $J_1$ depends only on $\|\varphi\|_\infty$, $\delta$ and $T$. By the compactness of $M$, we conclude that for $t \in [\delta, T]$, the value function $u(\cdot, t)$ is Lipschitz on $M$.

We are now going to show the locally Lipschitz continuity of $u(x,t)$ in $t$. Given $t$ and $t'$ with $\delta \le t < t' \le T$. Let $\gamma : [0, t'] \to M$ be a minimizer of $u(x, t')$, then

$$u(x,t') - u(x,t) = u(\gamma(t),t) - u(x,t) + \int_{t}^{t'} \Big[L(\gamma(s),\dot{\gamma}(s)) - \lambda(\gamma(s))u(\gamma(s),s)\Big]ds,$$

where the bound of $\|\dot{\gamma}(s)\|$ depends only on $\|\varphi\|_\infty$, $\delta$ and $T$. We have shown that for $t \ge \delta$, the following holds

$$u(\gamma(t),t) - u(x,t) \le J_1 d(\gamma(t), x) \le J_1 \int_{t}^{t'} \|\dot{\gamma}(s)\| ds \le J_2(t'-t).$$

Thus, $u(x,t') - u(x,t) \le J_3(t'-t)$, where $J_3$ depends only on $\|\varphi\|_\infty$, $\delta$ and $T$. The condition $t' < t$ is similar. We conclude the Lipschitz continuity of $u(x,\cdot)$ on $[\delta, T]$. □

Let $\|T_t^-\varphi(x)\|_\infty \le K$ for all $t \ge 0$, with the bound $K$ independent of $t$. Note that $T_t^-\varphi(x) = T_1^- \circ T_{t-1}^-\varphi(x)$. Fix $\delta = 1/2$ and $T = 1$ in Theorem A.9. It follows that the Lipschitz constant of $T_1^- \circ T_{t-1}^-\varphi(x)$ depends only on $K$, which is independent of $t$. This completes the proof of Proposition 1.6.

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