AUBRY-MATHER THEORY FOR CONTACT HAMILTONIAN SYSTEMS III

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Abstract. By exploiting the dynamics around the Aubry set of contact Hamiltonian systems, we provide a relation among the Mather set, the recurrent set, the strongly static set, the Aubry set, the Mañé set and the non-wandering set. Moreover, we consider the strongly static set, as a new flow-invariant set between the Mather set and the Aubry set, in the strictly increasing case. We show that this set plays an essential role in the representation of certain minimal forward weak KAM solution and the existence of transitive orbits around the Aubry set.

Contents

1. Introduction

In [13, 21], the Aubry-Mather theory was developed for conformally symplectic systems and contact Hamiltonian systems with strictly increasing dependence on the contact variable u respectively. The conformally symplectic systems are closely related to discounted Hamiltonian systems (see *e.g.*, [6, 16]), which serve as a class of typical examples for more general contact cases. In [23], the Aubry-Mather theory was further developed for contact Hamiltonian systems with non-decreasing dependence on u. More information on the Aubry set was founded, such as the comparison property, graph property and a partially ordered relation for the collection of all projected Aubry sets with respect to backward weak KAM solutions. Loosely speaking, the Aubry-Mather theory and weak KAM theory are two kinds of parallel ways to describe the global minimizing dynamics of contact Hamiltonian systems. The former is concerned with "orbits", while the later focus on "weak KAM solutions". This kind of solutions can be viewed as certain generalization of generating functions in Hamiltonian systems. One

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can also see [1, Section 46] for a vivid description on the connection between orbits and solutions of Hamilton-Jacobi equations.

1.1. **Basic assumptions.** Assume M is a connected, closed (compact without boundary) and smooth Riemannian manifold. We choose, once and for all, a C^∞ Riemannian metric g on M. Denote by $\mathrm{dist}(\cdot,\cdot)$ and $d(\cdot,\cdot)$ the distance on M and $T^*M\times\mathbb{R}$ induced by g respectively. $C(M,\mathbb{R})$ stands for the space of continuous functions on M. $\|\cdot\|_\infty$ denotes the supremum norm on $C(M,\mathbb{R})$. $\|\cdot\|_x$ denotes a norm on T^*_xM and T_xM. Let $H: T^*M\times\mathbb{R}\to\mathbb{R}$ be a C^3 function satisfying

- **(H1)** *Strict convexity*: $\frac{\partial^2 H}{\partial p^2}(x,p,u)$ is positive definite for all $(x,p,u)\in T^*M\times\mathbb{R}$;
- **(H2)** *Superlinearity*: for every $(x,u)\in M\times\mathbb{R}$, $H(x,p,u)$ is superlinear in p;
- **(H3)** *Non-decreasing*: there is a constant $\lambda>0$ such that for every $(x,p,u)\in T^*M\times\mathbb{R}$,

$$0\leq \frac{\partial H}{\partial u}(x,p,u)\leq\lambda.$$

We consider the contact Hamiltonian system generated by

$$\begin{cases} \dot{x}=\frac{\partial H}{\partial p}(x,p,u), \\ \dot{p}=-\frac{\partial H}{\partial x}(x,p,u)-\frac{\partial H}{\partial u}(x,p,u)p, \\ \dot{u}=\frac{\partial H}{\partial p}(x,p,u)\cdot p-H(x,p,u). \end{cases} \qquad (x,p,u)\in T^*M\times\mathbb{R}, \tag{CH}$$

In order to handle global dynamics, it is necessary to assume additionally

- **(A)** *Admissibility*: there exists $a\in\mathbb{R}$ such that

$$\inf_{u\in C^\infty(M,\mathbb{R})}\sup_{x\in M} H(x,Du,a)=0.$$

This formulation is inspired by the concept of the Mañé critical value [5]. From a PDE point of view, the assumption (A) holds true if and only if the stationary Hamilton-Jacobi equation

$$H(x,Du,u)=0,\quad x\in M,$$

has a viscosity solution (see [18, Theorem 1.4]). If H is independent of u, this equivalence was shown in [12].

The necessity of (A) can be shown by the following example:

$$H(x,p,u)=h(x,p)+g(x)u,\quad x\in\mathbb{T},$$

where $\mathbb{T}$ denotes a flat circle. The function $g:\mathbb{T}\to\mathbb{R}$ does not vanish identically and satisfies $0\leq g(x)\leq\lambda$. If $g(x)>0$ for all $x\in\mathbb{T}$, based on the compactness of $\mathbb{T}$, $g(x)\geq\delta$ for certain positive constant δ. In this case, $h(x,Du)+g(x)u=0$ has the unique viscosity solution. Namely, (A) always holds. If there exist $x_0\in\mathbb{T}$ such that $g(x_0)=0$, then (A) may not hold. For example, consider the Hamilton-Jacobi equation

$$\frac{1}{2}|Du|^2+V(x)+g(x)u=0.$$

Assume $V:\mathbb{T}\to\mathbb{R}$ is of class C^3 with $V(x_0)>0$ and $g(x_0)=0$. Then for all $a\in\mathbb{R}$,

$$\begin{aligned} &\inf_{u\in C^\infty(\mathbb{T},\mathbb{R})}\sup_{x\in\mathbb{T}}\left\{\frac{1}{2}|Du|^2+V(x)+g(x)a\right\} \\ \geq&\inf_{u\in C^\infty(\mathbb{T},\mathbb{R})}\left\{\frac{1}{2}|Du|^2+V(x_0)+g(x_0)a\right\} \\ =&V(x_0)>0. \end{aligned}$$

Therefore, (A) is necessary to be assumed.

1.2. **Aims, obstructions and contributions.** In this paper, we continue to develop the Aubry-Mather theory and weak KAM theory for contact Hamiltonian systems under (H1)-(H3) and (A). It is well known that the Aubry set plays a central role in both theories for classical Hamiltonian systems. In [15], R. Mañé obtained some properties of the Aubry set from the perspective of topological dynamics. Inspired by this work, we are concerned with the following problems.

- The topological dynamics on the Aubry set, such as the recurrence property, the non-wandering property and their relations to the Mather set and Mañé set.
- The representation of weak KAM solutions, and the interplay between weak KAM solutions and the dynamics around the Aubry set.

Under (H1)-(H3) and (A), the backward and forward weak KAM solutions are not one-to-one correspondent like classical cases (i.e. $\partial_u H \equiv 0$). We have to deal with some new issues as follows.

(1) In classical cases, the Aubry set is chain-recurrent (see [4, 15]). Even in strictly increasing contact cases, the Aubry set may contain non-chain recurrent points (see Proposition 2.11(i)(ii) below).
(2) In strictly increasing cases, the backward weak KAM solution is always unique. Unfortunately, the dynamics reflected by the backward weak KAM solution is too rough. Thus, one has to exploit the structure of the set of forward weak KAM solutions to reveal more dynamical information. However, the structure of this set is rather complicated (see Proposition 2.11(iii)(iv) below).
(3) The complicated structure of the set of weak KAM solutions causes certain difficulties to show the the interplay between weak KAM solutions and the dynamics around the Aubry set. For example, even if $\partial_u H$ vanishes at only one point, some new phenomena from both dynamical and PDE aspects would appear. More precisely, we consider
$$\frac{1}{2}|Du|^2 + f(x)u = 0, \quad x \in \mathbb{T},$$
where $f : \mathbb{T} \to \mathbb{R}$ is a C^3 function with $f(x_0) = 0$, $f(x) > 0$ for all $x \in \mathbb{T}\backslash\{x_0\}$. It is clear that $u \equiv 0$ is a viscosity solution. Besides, there exists an uncountable family of nontrivial viscosity solutions $\{v_i\}_{i\in I}$ and the definition of the Aubry set essentially depends on v_i (see [23, Proposition 1.11] for more details). Comparably, in classical cases, the definition of the Aubry set is independent of viscosity solutions.

Corresponding to the issues above, we summarize the main contributions in this paper as follows:

- Regarding Item (1), we find the Aubry set is too large to characterize the dynamics with recurrent or non-wandering property. Thus, we introduce so called the strongly static set, which is a new flow invariant subset of the Aubry set. We prove that the strongly static set is always non-wandering. Moreover, in order to locate this set in a series of action minimizing invariant sets, we prove an inclusion relation among the Mather set, the Aubry set, the Mañé set, the recurrent points and the non-wandering points. For the definitions of the first three sets, see Section 2.1.2 below. The latter two sets are the basic concepts in the classical theory of topological dynamical systems. This result is given by

Theorem 1. It is worth mentioning that the strongly static set always coincides with the Aubry set in the classical case.

- Regarding Item (2), we focus on the structure of the set of forward weak KAM solutions in the strictly increasing case. The existence of the *maximal* element in this set was shown in [21]. Unfortunately, the *minimal* forward weak KAM solution may not exist in the sense of total order. For example,

$$u + \frac{1}{2}|Du|^2 = 0, \quad x \in \mathbb{T}, \tag{1.1}$$

where $\mathbb{T} := (-\frac{1}{2}, \frac{1}{2}]$ denotes the flat unit circle. Let $\mathcal{S}_+$ be the set of all forward weak KAM solutions of (1.1). It is not difficult to see

$$v(x) := \min_{\mathcal{S}_+} v_+(x) \equiv -\frac{1}{8},$$

which is not a forward weak KAM solution of (1.1). Based on Zorn's lemma, we prove the existence of the *minimal* forward weak KAM solution in a partially ordered sense. Moreover, we show that the strongly static set plays an essential role in the representation of the *minimal* forward weak KAM solution. This result is given by Theorem 2. Loosely speaking, in the strictly increasing cases, the Aubry set is only related to the unique backward weak KAM solution and the maximal one. The strongly static set is necessarily involved in order to characterize the property of forward weak KAM solutions except the maximal one.

- Regarding Item (3), since the Aubry set may contain wandering points, we need to introduce a more flexible dynamics to detect the interplay between weak KAM solutions and the dynamics around the Aubry set. The non-wandering property can be viewed as "neighborhood recurrence". Thus, we consider a kind of dynamical property that can be viewed as "neighborhood transition". More precisely, we introduce the following definition.

Definition 1.1 (transitive orbit)**.** *Given $X_1, X_2 \in T^*M \times \mathbb{R}$, we say there is a transitive orbit from X_1 to X_2 if for any neighborhoods U_1 of X_1 and U_2 of X_2, there exists an orbit that begins in U_1 and later passes through U_2.*

Remark 1.2. *It is clear that a transitive orbit from X_1 to X_2 is a pseudo-orbit with arbitrarily small jumps. In particular, there are at most two jumps of the transitive orbit, and these jumps are only allowed to happen around the adjoining points of X_1 and X_2. Following [17, Definition 1.1.8], we write $X_1 \rightsquigarrow X_2$ in the following if there is a transitive orbit from X_1 to X_2. Similar to [17, Lemma 1.1.2], the relation*

$$\mathscr{R}(\rightsquigarrow) = \{(X_1, X_2) \in (T^*M \times \mathbb{R})^2 : \ X_1 \rightsquigarrow X_2\}$$

is also closed, see the proof below Lemma 4.2 in Section 4.

Finally, we obtain a result on the interplay among weak KAM solutions, the strongly static set and the existence of transitive orbits around the Aubry set. This result is given by Theorem 3.

2. Statement of main results

To state the main results (Theorem 1, Theorem 2 and Theorem 3 below) in a precise way, we need to prepare some notions and notations. They mainly come from [19–23].

2.1. **Notions and notations.**

2.1.1. *Weak KAM solutions.* Let $L : TM\times\mathbb{R} \to \mathbb{R}$ be the contact Lagrangian associated to $H(x, p, u)$ via

$$L(x, \dot{x}, u) := \sup_{p\in T_x^*M} \{\langle \dot{x}, p\rangle_x - H(x, p, u)\},$$

where $\langle\cdot,\cdot\rangle_x$ represents the canonical pairing between the tangent and cotangent space at $x \in M$. Since H satisfies (H1), (H2) and (H3), then $L(x, \dot{x}, u)$ satisfies

- **(L1)** *Strict convexity*: $\frac{\partial^2 L}{\partial \dot{x}^2}(x, \dot{x}, u)$ is positive definite for all $(x, \dot{x}, u) \in TM \times \mathbb{R}$;
- **(L2)** *Superlinearity*: for every $(x, u) \in M \times \mathbb{R}$, $L(x, \dot{x}, u)$ is superlinear in $\dot{x}$;
- **(L3)** *Non-increasing*: there is a constant $\lambda > 0$ such that for every $(x, \dot{x}, u) \in TM\times\mathbb{R}$,

$$-\lambda \le \frac{\partial L}{\partial u}(x, \dot{x}, u) \le 0.$$

Following Fathi [8], one can define weak KAM solutions of

$$H(x, Du, u) = 0, \quad x \in M. \tag{HJ}$$

It can be proved that the backward weak KAM solutions of (HJ) are equivalent to the viscosity solutions.

Definition 2.1. *A function $u_- \in C(M,\mathbb{R})$ is called a backward weak KAM solution of* (HJ) *if*

(i) *for each continuous piecewise C^1 curve $\gamma : [t_1, t_2] \to M$, we have*

$$u_-(\gamma(t_2)) - u_-(\gamma(t_1)) \le \int_{t_1}^{t_2} L(\gamma(s), \dot{\gamma}(s), u_-(\gamma(s)))ds;$$

(ii) *for each $x \in M$, there exists a C^1 curve $\gamma : (-\infty, 0] \to M$ with $\gamma(0) = x$ such that*

$$u_-(x) - u_-(\gamma(t)) = \int_t^0 L(\gamma(s), \dot{\gamma}(s), u_-(\gamma(s)))ds, \quad \forall t < 0. \tag{2.1}$$

Similarly, a function $u_+ \in C(M,\mathbb{R})$ is called a forward weak KAM solution of (HJ) *if it satisfies (i) and for each $x \in M$, there exists a C^1 curve $\gamma : [0, +\infty) \to M$ with $\gamma(0) = x$ such that*

$$u_+(\gamma(t)) - u_+(x) = \int_0^t L(\gamma(s), \dot{\gamma}(s), u_+(\gamma(s)))ds, \quad \forall t > 0. \tag{2.2}$$

We denote by $\mathcal{S}_-$ (resp. $\mathcal{S}_+$) the set of backward (resp. forward) weak KAM solutions of equation (HJ).

2.1.2. *Action minimizing objects.* The definitions of the action minimizing invariant sets are based on the variational principle of contact Hamiltonian systems. See [19, Theorem A] for the following result, which holds under (H1), (H2) and $|\frac{\partial H}{\partial u}| \le \lambda$ instead of (H3).

Proposition 2.2. *For any given $x_0 \in M$, $u_0 \in \mathbb{R}$, there exists a continuous function $h_{x_0,u_0}(x, t)$ defined on $M \times (0, +\infty)$ satisfying*

$$h_{x_0,u_0}(x, t) = u_0 + \inf_{\substack{\gamma(0)=x_0\\ \gamma(t)=x}} \int_0^t L(\gamma(\tau), \dot{\gamma}(\tau), h_{x_0,u_0}(\gamma(\tau), \tau))d\tau, \tag{2.3}$$

where the infimum is taken among Lipschitz continuous curves $\gamma:[0,t]\to M$. Moreover, the infimum in (2.3) is achieved. Let γ be a Lipschitz curve achieving the infimum and

$$x(s):=\gamma(s),\ u(s):=h_{x_0,u_0}(\gamma(s),s),\ p(s):=\frac{\partial L}{\partial \dot{x}}(\gamma(s),\dot{\gamma}(s),u(s)).$$

Then $(x(\cdot),p(\cdot),u(\cdot)):[0,t]\to T^*M\times\mathbb{R}$ satisfies equations (CH) with $x(0)=x_0$, $x(t)=x$ and

$$\lim_{s\to 0^+} u(s)=u_0.$$

We associate to $h_{x_0,u_0}(x,t)$ another action function $h^{x_0,u_0}(x,t)$, which is also defined implicitly by

$$h^{x_0,u_0}(x,t)=u_0-\inf_{\substack{\gamma(t)=x_0\\ \gamma(0)=x}}\int_0^t L(\gamma(\tau),\dot{\gamma}(\tau),h^{x_0,u_0}(\gamma(\tau),t-\tau))d\tau, \tag{2.4}$$

where the infimum is taken among Lipschitz continuous curves $\gamma:[0,t]\to M$.

Based on the action functions, one can define action minimizing curves.

Definition 2.3 (Globally minimizing curves)**.** *A curve $(x(\cdot),u(\cdot)):\mathbb{R}\to M\times\mathbb{R}$ is called globally minimizing, if it is locally Lipschitz and for each t_1, $t_2\in\mathbb{R}$ with $t_1<t_2$, there holds*

$$u(t_2)=h_{x(t_1),u(t_1)}(x(t_2),t_2-t_1). \tag{2.5}$$

The positively minimizing curves (resp. negatively minimizing curves) can be defined in a similar manner. We say positively (resp. negatively), we mean the curve is defined on $\mathbb{R}_+$ (resp. $\mathbb{R}_-$), and (2.5) holds for t_1, $t_2\in\mathbb{R}_+$ (resp. $\in\mathbb{R}_-$). If a curve $(x(\cdot),u(\cdot)):\mathbb{R}\to M\times\mathbb{R}$ is global minimizing, then $x:\mathbb{R}\to M$ is of class C^1. Let

$$p(t):=\frac{\partial L}{\partial \dot{x}}(x(t),\dot{x}(t),u(t)),\quad t\in\mathbb{R}. \tag{2.6}$$

Then $(x(\cdot),p(\cdot),u(\cdot)):\mathbb{R}\to T^*M\times\mathbb{R}$ satisfies equations (CH) (see [21, Propostion 3.1]). Following Mañé [15], the notion of static and semi-static curves for contact Hamiltonian systems were introduced in [21] and [23] respectively.

Definition 2.4 (Semi-static curves)**.** *A curve $(x(\cdot),u(\cdot)):\mathbb{R}\to M\times\mathbb{R}$ is called semi-static, if it is globally minimizing and for each $t_1\le t_2\in\mathbb{R}$, there holds*

$$u(t_2)=\inf_{s>0} h_{x(t_1),u(t_1)}(x(t_2),s). \tag{2.7}$$

Definition 2.5 (Semi-static orbits)**.** *If a curve $(x(\cdot),u(\cdot)):\mathbb{R}\to M\times\mathbb{R}$ is semi-static , then $(x(\cdot),p(\cdot),u(\cdot)):\mathbb{R}\to T^*M\times\mathbb{R}$ satisfies equations (CH), where $p(\cdot)$ is determined by (2.6). We call it a semi-static orbit.*

The positively (resp. negatively) semi-static orbits can be also defined in a similar manner. Denote the flow generated by (CH) by Φ_t. We define some flow invariant sets as follows.

Definition 2.6 (Mañé set)**.** *We call the set of all semi-static orbits the Mañé set for H, denoted by $\tilde{\mathcal{N}}$.*

We call $\mathcal{N}:=\pi^*\tilde{\mathcal{N}}$ the projected Mañé set. We denote, once and for all

$$\pi^*:T^*M\times\mathbb{R}\to M.$$

We define $\tilde{\mathcal{N}}^+$ (resp. $\tilde{\mathcal{N}}^-$) as the set of all positively (resp. negatively) semi-static orbits.

Definition 2.7 (Static curves)**.** *A curve* $(x(\cdot), u(\cdot)) : \mathbb{R} \to M \times \mathbb{R}$ *is called static, if it is globally minimizing and for each* $t_1, t_2 \in \mathbb{R}$*, there holds*

$$u(t_2) = \inf_{s>0} h_{x(t_1),u(t_1)}(x(t_2), s) \tag{2.8}$$

A *static orbit* is defined as $(x(\cdot), p(\cdot), u(\cdot)) : \mathbb{R} \to T^*M \times \mathbb{R}$, where $p(\cdot)$ is determined by (2.6).

Definition 2.8 (Aubry set)**.** *We call the set of all static orbits the Aubry set for* H*, denoted by* $\tilde{\mathcal{A}}$*. The Aubry set is also called the static set.*

We call $\mathcal{A} := \pi^*\tilde{\mathcal{A}}$ the projected Aubry set. Inspired by Mather [14], we define a subset of the Aubry set from a measure theoretic point of view, so called the Mather set. Based on Proposition 3.9 below, there exist Borel Φ_t-invariant probability measures supported in $\tilde{\mathcal{N}}$, called *Mather measures.* Denote by $\mathfrak{M}$ the set of Mather measures. The *Mather set* of contact Hamiltonian systems (CH) is defined by

$$\tilde{\mathcal{M}} = \mathrm{cl}\left(\bigcup_{\mu\in\mathfrak{M}} \mathrm{supp}(\mu)\right), \tag{2.9}$$

where $\mathrm{supp}(\mu)$ denotes the support of μ.

The invariance of these sets above follows directly from their definitions.

2.1.3. *Strongly static set.* If H is independent of u, the Aubry set is chain-recurrent. Unfortunately, it is not true in general contact settings. In order to characterize the chain-recurrence in the Aubry set, we introduce a new flow invariant set, called strongly static set.

Definition 2.9 (Strongly static curves)**.** *A curve* $(x(\cdot), u(\cdot)) : \mathbb{R} \to M \times \mathbb{R}$ *is called strongly static, if it is globally minimizing and for each* $t_1, t_2 \in \mathbb{R}$*, there holds*

$$u(t_2) = \sup_{s>0} h^{x(t_1),u(t_1)}(x(t_2), s). \tag{2.10}$$

A *strongly static orbit* is defined as $(x(\cdot), p(\cdot), u(\cdot)) : \mathbb{R} \to T^*M \times \mathbb{R}$, where $p(\cdot)$ is determined by (2.6).

Definition 2.10 (Strongly static set)**.**

$$\tilde{\mathcal{S}}_s := \mathrm{cl}(\{\textit{all strongly static orbits}\}).$$

where $\mathrm{cl}(A)$ *denotes the closure of* $A \subseteq T^*M \times \mathbb{R}$*.*

The differences between $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{S}}_s$ are shown by the following Proposition 2.11. For the consistency, we postpone its proof in Appendix B.

Proposition 2.11. *Let* $\lambda > 0$ *and*

$$H(x, p, u) := \lambda u + \frac{1}{2}|p|^2 + p \cdot V(x), \quad x \in T, \tag{E}$$

where $\mathbb{T}$ *denotes a flat circle and* $V : \mathbb{T} \to \mathbb{R}$ *is a* C^3 *function which has exactly two vanishing points* x_1*,* x_2 *with* $V'(x_1) > 0$*,* $V'(x_2) < 0$*. Let* $\mathcal{S}_-$ *and* $\mathcal{S}_+$ *be the set of the backward and forward weak KAM solutions of (E) respectively. Then* $u_- \equiv 0$ *is the unique element in* $\mathcal{S}_-$*. Thus,* $u_+ \equiv 0 \in \mathcal{S}_+$*. Moreover,*

(i) *if* $\lambda > |V'(x_2)|$*, then the point* $(x_2, 0, 0)$ *is a sink in* $T^*\mathbb{T} \times \mathbb{R}$*;*

(ii) *for any* $\lambda > 0$,

$$\tilde{\mathcal{A}} = \{(x, 0, 0) \mid x \in \mathbb{T}\}, \quad \tilde{\mathcal{S}}_s = \{(x_1, 0, 0), \ (x_2, 0, 0)\};$$

(iii) *if* $\lambda < |V'(x_2)|$, *the set* $\mathcal{S}_+$ *consists of two elements* $u_+ \equiv 0$ *and* w_+, *where* $w_+ : \mathbb{T} \to \mathbb{R}$ *satisfies* $w_+(x_1) = 0$, $w_+(x) < 0$ *for each* $x \in \mathbb{T}\backslash\{x_1\}$*;*

(iv) *for* λ *large enough,* $\mathcal{S}_+$ *may contains more than two elements.*

By Proposition 2.11(i)(ii), $\tilde{\mathcal{A}}$ contains non-chain recurrent points in the example (E). Nevertheless, $\tilde{\mathcal{S}}_s$ is non-wandering. For Item (iii), we have a rough picture for $\mathcal{S}_\pm$ with $V(x) = \sin x$ (see Figure 1).

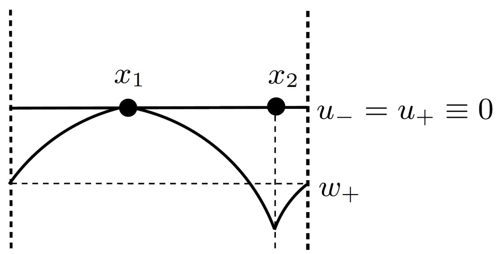

FIGURE 1. $\mathcal{S}_\pm$ in Item (iii)

2.2. **Main results.** First of all, we locate the strongly static set in a series of action minimizing invariant sets, and show its relations to the recurrence property and the non-wandering property.

Theorem 1 (**Topological dynamics around the Aubry set**). *Let* $\tilde{\mathcal{R}}$ *be the set of recurrent points. Let* $\tilde{\Omega}$ *be the set of non-wandering points. Then*

$$\emptyset \neq \tilde{\mathcal{M}} \subseteq \tilde{\mathcal{N}} \cap \mathrm{cl}(\tilde{\mathcal{R}}) \subseteq \tilde{\mathcal{S}}_s \subseteq \tilde{\mathcal{A}} \cap \tilde{\Omega}.$$

The closure $\mathrm{cl}(\tilde{\mathcal{R}})$ is called the Birkhoff center. The fact $\tilde{\mathcal{M}} \subseteq \tilde{\mathcal{N}} \cap \mathrm{cl}(\tilde{\mathcal{R}})$ follows easily from the Poincaré recurrence theorem. To prove $\tilde{\mathcal{N}} \cap \mathrm{cl}(\tilde{\mathcal{R}}) \subseteq \tilde{\mathcal{S}}_s$, we need to establish the Lipschitz continuity of the Mañé potentials, whose proof is postponed to Appendix A.2. The inclusion $\tilde{\mathcal{S}}_s \subseteq \tilde{\mathcal{A}} \cap \tilde{\Omega}$ follows from a technical lemma on transitive criterion (see Lemma 4.1 below).

In order to show the differences of the dynamics between the classical cases and the contact cases, we enhance the assumption (H3) by

(H3') *Strictly increasing*: there is a constant $\lambda > 0$ such that for every $(x, p, u) \in T^*M \times \mathbb{R}$,

$$0 < \frac{\partial H}{\partial u}(x, p, u) \leq \lambda,$$

Under (H1), (H2), (H3') and (A), it is well known that the set of backward weak KAM solutions $\mathcal{S}_-$ consists of only one element. Consequently, the Mañé set coincides with the Aubry set (see [23, Remark 2]). However, the structure of the set of forward weak KAM solutions $\mathcal{S}_+$ may be rather complicated. That causes significant differences between the Aubry set and the strongly static set, as it is shown in Proposition 2.11.

In order to deal with the other elements in $\mathcal{S}_+$ except the maximal one. We define a partial order in $\mathcal{S}_+$:

$$v_1 \preceq v_2 \text{ if and only if } v_1(x) \leq v_2(x) \text{ for all } x \in M.$$

Moreover, we define $\mathcal{Z}_{\max}$ a maximal totally ordered subset of $\mathcal{S}_+$. Namely, for any $w_+ \in \mathcal{S}_+ \backslash \mathcal{Z}_{\max}$, there exist $v_+ \in \mathcal{Z}_{\max}$ and $x_1, x_2 \in M$ such that $w_+(x_2) > v_+(x_2)$ and $w_+(x_2) < v_+(x_2)$. We will show the existence of minimal elements in $\mathcal{S}_+$ in this sense of partial order. Morever, we will provide a representation for the minimal element in $\mathcal{Z}_{\max}$.

Theorem 2 (**Minimal forward weak KAM solutions).**

(1) *The partially ordered set $(\mathcal{S}_+, \preceq)$ has minimal elements.*
(2) *For each $\mathcal{Z}_{\max} \subseteq \mathcal{S}_+$, there exists $x_0 \in \mathcal{M}$ depending on $\mathcal{Z}_{\max}$, such that the minimal element u^* in $\mathcal{Z}_{\max}$ can be represented in the following two manners:*

$$u^*(x) = \inf_{\substack{v_+(x_0)=u_-(x_0) \\ v_+ \in \mathcal{S}_+}} v_+(x) = \limsup_{t \to +\infty} h^{x_0, u_-(x_0)}(x, t),$$

where $\mathcal{M}$ denotes the projected Mather set and $h^{\cdot,\cdot}(\cdot,\cdot) : M \times \mathbb{R} \times M \times \mathbb{R}_+ \to \mathbb{R}$ is the action function defined by (2.4).

See Remark 6.8 below for a discussion on the choice of x_0, from which one can see that the strongly static set plays an essential role. By analysing more detailed structure of $\mathcal{S}_+$, one has

Theorem 3 (**Existence of transitive orbits).** *Given $X_1 := (x_1, p_1, u_1) \in \tilde{\mathcal{A}}$, $X_2 := (x_2, p_2, u_2) \in \tilde{\mathcal{S}}_s$, if for each $v_+ \in \mathcal{S}_+$, $v_+(x_2) = u_-(x_2)$ implies $v_+(x_1) = u_-(x_1)$, then $X_1 \rightsquigarrow X_2$.*

If the forward weak KAM solution is unique, $\tilde{\mathcal{A}} = \tilde{\mathcal{S}}_s$ (see [23, Proposition 10]). In this case, the projected strongly static set can be characterized as follows

$$\mathcal{S}_s = \{x \in M \mid u_-(x) = u_+(x)\},$$

where u_- (resp. u_+) denotes the unique backward (resp. forward) weak KAM solution. From Theorem 3, we have the following corollary.

Corollary 2.12. *Given any two points $X_1, X_2 \in \tilde{\mathcal{A}}$, if the forward weak KAM solution is unique, then $X_1 \rightsquigarrow X_2$.*

Figure 2 provides a rough picture for the dynamics around $\tilde{\mathcal{A}}$ (projected to $M \times \mathbb{R}$) under the assumption of Theorem 3, where

- Γ_c denotes a transitive orbit from X_1 to X_2;
- Γ_n^1, Γ_n^2 denote non-wandering orbits to X_1 and X_2 respectively.

The rest of this paper is organized as follows. In Section 3, we recall some useful facts, which mainly come from [21, 23]. In Section 4, we prove a technical lemma on the existence of certain transitive orbit, from which we obtain the topological dynamics around the Aubry set in Section 5. In Section 6, we provide more detailed information on the set of forward weak KAM solutions in the strictly increasing cases. Some auxiliary results are proved in Appendix A. Finally, we provide a proof of Proposition 2.11 in Appendix B.

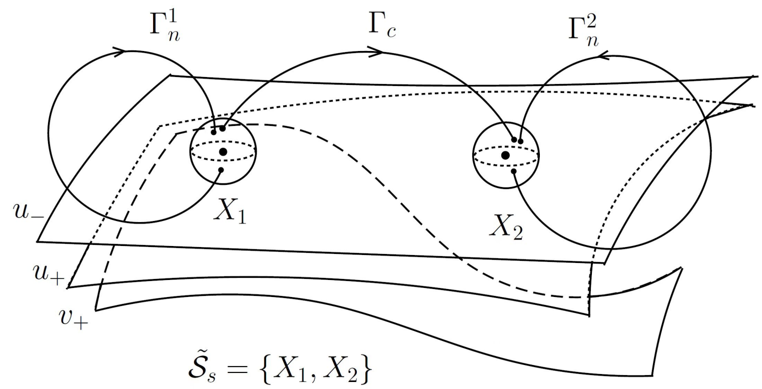

FIGURE 2. The dynamics around the Aubry set

3. Preliminaries

Recently, a variational approach for contact Hamiltonian systems was developed in a series of papers [18–24]. Different from classical cases, all of technical tools for contact Hamiltonian systems were formed in an implicit manner. The main reason to use the implicit form is to get rid of the constraints caused by the u-argument. Besides, it is worth mentioning that an alternative variational formulation was provided in [2, 11] in light of G. Herglotz's work [9].

In the following, we collect some facts used in this paper. All of these results hold under (H1), (H2) and $|\frac{\partial H}{\partial u}| \leq \lambda$.

3.1. Action functions and minimizing curves.

Let us collect some properties of the action functions $h_{x_0,u_0}(x,t)$ and $h^{x_0,u_0}(x,t)$ in the following propositions. See [19, Theorems C, D] and [20, Theorem 3.1, Propositions 3.1-3.4] for more details.

Proposition 3.1.

(1) *(Monotonicity). Given $x_0 \in M$, u_0, u_1, $u_2 \in \mathbb{R}$, if $u_1 < u_2$, then $h_{x_0,u_1}(x,t) < h_{x_0,u_2}(x,t)$, for all $(x,t) \in M \times (0,+\infty)$;*

(2) *(Minimality). Given x_0, $x \in M$, $u_0 \in \mathbb{R}$ and $t > 0$, let $S^{x,t}_{x_0,u_0}$ be the set of the solutions $(x(s), p(s), u(s))$ of (CH) on $[0,t]$ with $x(0) = x_0$, $x(t) = x$, $u(0) = u_0$. Then*

$$h_{x_0,u_0}(x,t) = \inf\{u(t) \mid (x(s), p(s), u(s)) \in S^{x,t}_{x_0,u_0}\}, \quad \forall (x,t) \in M \times (0,+\infty).$$

(3) *(Lipschitz continuity). The function $(x_0, u_0, x, t) \mapsto h_{x_0,u_0}(x,t)$ is locally Lipschitz continuous on $M \times \mathbb{R} \times M \times (0,+\infty)$.*

(4) *(Markov property). Given $x_0 \in M$, $u_0 \in \mathbb{R}$,*

$$h_{x_0,u_0}(x,t+s) = \inf_{y \in M} h_{y,h_{x_0,u_0}(y,t)}(x,s)$$

for all s, $t > 0$ and all $x \in M$. Moreover, the infimum is attained at y if and only if there exists a minimizer γ of $h_{x_0,u_0}(x,t+s)$ with $\gamma(t) = y$.

(5) *(Reversibility). Given x_0, $x \in M$ and $t > 0$, for each $u \in \mathbb{R}$, there exists a unique $u_0 \in \mathbb{R}$ such that*

$$h_{x_0,u_0}(x,t) = u.$$

Proposition 3.2.

(1) *(Monotonicity). Given* $x_0 \in M$ *and* u_1, $u_2 \in \mathbb{R}$*, if* $u_1 < u_2$*, then* $h^{x_0,u_1}(x,t) < h^{x_0,u_2}(x,t)$*, for all* $(x,t) \in M \times (0,+\infty)$*;*
(2) *(Maximality). Given* x_0, $x \in M$, $u_0 \in \mathbb{R}$ *and* $t > 0$*, let* $S^{x_0,u_0}_{x,t}$ *be the set of the solutions* $(x(s), p(s), u(s))$ *of (CH) on* $[0,t]$ *with* $x(0) = x$, $x(t) = x_0$, $u(t) = u_0$*. Then*

$$h^{x_0,u_0}(x,t) = \sup\{u(0) \mid (x(s), p(s), u(s)) \in S^{x_0,u_0}_{x,t}\}, \quad \forall (x,t) \in M \times (0,+\infty).$$

(3) *(Lipschitz continuity). The function* $(x_0, u_0, x, t) \mapsto h^{x_0,u_0}(x,t)$ *is locally Lipschitz continuous on* $M \times \mathbb{R} \times M \times (0,+\infty)$*.*
(4) *(Markov property). Given* $x_0 \in M$, $u_0 \in \mathbb{R}$*,*

$$h^{x_0,u_0}(x, t+s) = \sup_{y \in M} h^{y, h^{x_0,u_0}(y,t)}(x,s)$$

for all s, $t > 0$ *and all* $x \in M$*. Moreover, the supremum is attained at* y *if and only if there exists a minimizer* γ *of* $h^{x_0,u_0}(x,t+s)$*, such that* $\gamma(t) = y$*.*
(5) *(Reversibility). Given* x_0, $x \in M$*, and* $t > 0$*, for each* $u \in \mathbb{R}$*, there exists a unique* $u_0 \in \mathbb{R}$ *such that*

$$h^{x_0,u_0}(x,t) = u.$$

By definition, we have

Proposition 3.3. *Let* $(x(\cdot), u(\cdot)) : \mathbb{R} \to M \times \mathbb{R}$ *be a globally minimizing curve. Then for all* t_1, $t_2 \in \mathbb{R}$ *with* $t_1 \leq t_2$*,*

$$u(t_2) = \inf_{s>0} h_{x(t_1),u(t_1)}(x(t_2), s) \quad \textit{if and only if} \quad u(t_1) = \sup_{s>0} h^{x(t_2),u(t_2)}(x(t_1), s).$$

Remark 3.4. *By Proposition 3.3,*

- *a curve* $(x(\cdot), u(\cdot)) : \mathbb{R} \to M \times \mathbb{R}$ *is globally minimizing if and only if for each* $t_1 < t_2 \in \mathbb{R}$*,*
$$u(t_1) = h^{x(t_2),u(t_2)}(x(t_1), t_2 - t_1); \tag{3.1}$$
- *a curve* $(x(\cdot), u(\cdot)) : \mathbb{R} \to M \times \mathbb{R}$ *is semi-static if and only if it is globally minimizing and for each* $t_1 \leq t_2 \in \mathbb{R}$*,*
$$u(t_1) = \sup_{s>0} h^{x(t_2),u(t_2)}(x(t_1), s); \tag{3.2}$$
- *a positively (resp. negatively) semi-static curve can be also characterized in a similar manner.*

3.2. **Lax-Oleinik semigroups, weak KAM solutions and the Mañé set.** Let us recall two semigroups of operators introduced in [20]. Define a family of nonlinear operators $\{T^-_t\}_{t \geq 0}$ from $C(M,\mathbb{R})$ to itself as follows. For each $\varphi \in C(M,\mathbb{R})$, denote by $(x,t) \mapsto T^-_t\varphi(x)$ the unique continuous function on $M \times [0,+\infty)$ such that

$$T^-_t\varphi(x) = \inf_{\gamma}\left\{\varphi(\gamma(0)) + \int_0^t L(\gamma(\tau), \dot{\gamma}(\tau), T^-_\tau\varphi(\gamma(\tau)))d\tau\right\},$$

where the infimum is taken among absolutely continuous curves $\gamma : [0,t] \to M$ with $\gamma(t) = x$. Let γ be a curve achieving the infimum, and $x(s) := \gamma(s)$, $u(s) := T^-_s\varphi(x(s))$, $p(s) := \frac{\partial L}{\partial \dot{x}}(x(s), \dot{x}(s), u(s))$. Then $(x(s), p(s), u(s))$ satisfies (CH) with $x(t) = x$.

It is not difficult to see that $\{T^-_t\}_{t \geq 0}$ is a semigroup of operators and $T^-_t\varphi(x)$ is a viscosity solution of $w_t + H(x, w, w_x) = 0$ with $w(x,0) = \varphi(x)$.

Similarly, one can define another semigroup of operators $\{T_t^+\}_{t\geq 0}$ by

$$T_t^+\varphi(x) = \sup_{\gamma}\left\{\varphi(\gamma(t)) - \int_0^t L(\gamma(\tau),\dot\gamma(\tau),T_{t-\tau}^+\varphi(\gamma(\tau)))d\tau\right\},$$

where the infimum is taken among absolutely continuous curves $\gamma : [0,t] \to M$ with $\gamma(0) = x$. Let γ be a curve achieving the infimum, and $x(s) := \gamma(s)$, $u(s) := T_{t-s}^+\varphi(x(s))$, $p(s) := \frac{\partial L}{\partial \dot x}(x(s),\dot x(s),u(s))$. Then $(x(s),p(s),u(s))$ satisfies (CH) with $x(0) = x$.

The following proposition gives a relation between Lax-Oleinik semigroups and action functions. See [20, Propositions 4.1, 4.2] for details.

Proposition 3.5. *For each $\varphi \in C(M,\mathbb{R})$, we have*

$$T_t^-\varphi(x) = \inf_{y\in M} h_{y,\varphi(y)}(x,t), \quad T_t^+\varphi(x) = \sup_{y\in M} h^{y,\varphi(y)}(x,t), \quad \forall (x,t)\in M\times(0,+\infty).$$

The following proposition gives a relation between Lax-Oleinik semigroups and weak KAM solutions. See [18, Lemmas 4.1, 4.2, 6.2] for details.

Proposition 3.6. *The backward weak KAM solutions of (HJ) are the same as the viscosity solutions of (HJ). Moreover,*

- (i) *A function $u : M \to \mathbb{R}$ is a backward weak KAM solution of (HJ) if and only if $T_t^- u = u$ for all $t \geq 0$;*
- (ii) *A function $v : M \to \mathbb{R}$ is a forward weak KAM solution of (HJ) if and only if $T_t^+ v = v$ for all $t \geq 0$.*

By [21, Theorem 1.2] and [22, Theorem 1], we have

Proposition 3.7. *$\mathcal{S}_- \neq \emptyset$ if and only if $\mathcal{S}_+ \neq \emptyset$. More precisely, the following statements hold.*

- (1) *Let $v_- \in \mathcal{S}_-$. Then the function $x \mapsto \lim_{t\to\infty} T_t^+ v_-(x)$ is well defined, and it belongs to $\mathcal{S}_+$.*
- (2) *Let $v_+ \in \mathcal{S}_+$. Then the function $x \mapsto \lim_{t\to\infty} T_t^- v_+(x)$ is well defined, and it belongs to $\mathcal{S}_-$.*

In the following context of this section, we proceed under the assumption $\mathcal{S}_- \neq \emptyset$. It is well known that each $u_- \in \mathcal{S}_-$ is semiconcave and $u_+ \in \mathcal{S}_+$ is semiconvex. For each $u_\pm \in \mathcal{S}_\pm$, we define two subsets of $T^*M \times \mathbb{R}$ associated with $u_\pm$ respectively by

$$G_{u_\pm} := \mathrm{cl}\left(\{(x,p,u) \mid Du_\pm(x) \text{ exists},\ u = u_\pm(x),\ p = Du_\pm(x)\}\right), \tag{3.3}$$

where $\mathrm{cl}(A)$ denotes the closure of $A \subseteq T^*M \times \mathbb{R}$. Define

$$\tilde{\mathcal{N}}_{v_\pm} := \tilde{\mathcal{N}} \cap G_{v_\pm}, \quad \mathcal{N}_{v_\pm} := \pi^*\tilde{\mathcal{N}}_{v_\pm}.$$

The following proposition shows a relation between weak KAM solutions and semi-static curves. See [23, Proposition 17] for details.

Proposition 3.8. *Let $(x(\cdot),u(\cdot)) : \mathbb{R} \to M \times \mathbb{R}$ be a semi-static curve. Then there exists $v_- \in \mathcal{S}_-$ (resp. $v_+ \in \mathcal{S}_+$) such that $u(t) = v_-(x(t))$ (resp. $u(t) = v_+(x(t))$) for each $t \in \mathbb{R}$.*

The following proposition shows a relation between weak KAM solutions and the Mañé set. See [23, Theorem 1.3] for details.

Proposition 3.9. *Let $v_- \in \mathcal{S}_-$, $v_+ \in \mathcal{S}_+$. Let*

$$\mathcal{I}_{v_-} := \{x \in M \mid v_-(x) = \lim_{t\to\infty} T_t^+ v_-(x)\}$$

$$\mathcal{I}_{v_+} := \{x \in M \mid v_+(x) = \lim_{t\to\infty} T_t^- v_+(x)\}.$$

Then both $\mathcal{I}_{v_-}$ and $\mathcal{I}_{v_+}$ are not empty. Moreover,

$$\tilde{\mathcal{N}}_{v_\pm} = \{(x,p,u) \in T^*M \times \mathbb{R} \mid x \in \mathcal{I}_{v_\pm},\ u = v_\pm(x),\ p = Dv_\pm(x)\},$$

$$\tilde{\mathcal{N}} = \cup_{v_-\in\mathcal{S}_-}\tilde{\mathcal{N}}_{v_-} = \cup_{v_+\in\mathcal{S}_+}\tilde{\mathcal{N}}_{v_+}.$$

4. A technical lemma

In this section, we are devoted to proving a technical lemma on the existence of certain transitive orbit. It will be used in the proofs of Theorem 1 and Theorem 3.

Lemma 4.1 (Transitive criterion)**.** *Given $X_1 := (x_1, p_1, u_1) \in \tilde{\mathcal{A}}$, $X_2 := (x_2, p_2, u_2) \in \tilde{\mathcal{S}}_s$. If*

$$\lim_{t\to+\infty} h_{x_1,u_1}(x_2, t) = u_2,\ \limsup_{t\to+\infty} h^{x_2,u_2}(x_1, t) = u_1, \qquad (\diamondsuit)$$

then $X_1 \rightsquigarrow X_2$.

Let $\tilde{\mathcal{V}}$ be the set of $(x,p,u) \in T^*M \times \mathbb{R}$, for which there exists a strongly static orbit

$$(x(\cdot), p(\cdot), u(\cdot)) : \mathbb{R} \to T^*M \times \mathbb{R}$$

passing through (x,p,u). Let $\mathcal{V} = \pi^*\tilde{\mathcal{V}}$. By the definition of the strongly static set,

$$\tilde{\mathcal{S}}_s = \mathrm{cl}(\tilde{\mathcal{V}}), \quad \mathcal{S}_s = \mathrm{cl}(\mathcal{V}).$$

To prove Lemma 4.1, we only need to verify

Lemma 4.2. *Given any $X_1 := (x_1, p_1, u_1) \in \tilde{\mathcal{A}}$, $X_2 := (x_2, p_2, u_2) \in \tilde{\mathcal{V}}$. If*

$$\lim_{t\to+\infty} h_{x_1,u_1}(x_2, t) = u_2,\ \limsup_{t\to+\infty} h^{x_2,u_2}(x_1, t) = u_1,$$

then $X_1 \rightsquigarrow X_2$.

We give a proof that Lemma 4.2 implies Lemma 4.1.

Proof. Let $B(X,R)$ stand for the open metric ball on $T^*M \times \mathbb{R}$ centered at X with radius R, and let $\bar{B}(X,R)$ stand for its closure.

Given any $X_2 := (x_2, p_2, u_2) \in \tilde{\mathcal{S}}_s$. For any neighborhood U of X_2, one can find $R > 0$ such that $\bar{B}(X_2, R) \subset U$. Note that $\tilde{\mathcal{S}}_s = \mathrm{cl}(\tilde{\mathcal{V}})$. Thus, there exists a sequence $\{Z_n\}_{n\in\mathbb{N}} \subseteq \tilde{\mathcal{V}}$ such that

$$Z_n \to X_2, \quad n \to \infty.$$

Hence, there exists $N := N(R) > 0$ such that

$$d(X_2, Z_N) \le \frac{R}{4},$$

which implies

$$B\left(Z_N, \frac{R}{4}\right) \subset B(X_2, R).$$

By Definition 1.1, the existence of the transitive orbit from X_1 to Z_N implies $X_1 \rightsquigarrow X_2$. $\square$

The following lemma gives a way to obtain one-sided semi-static curves from one-sided minimizing curves.

Lemma 4.3.

(1) *Given* $(x_0, u_0) \in M \times \mathbb{R}$*, let* $(x(\cdot), u(\cdot)) : \mathbb{R}_+ \to M \times \mathbb{R}$ *be a positively minimizing curve with* $(x(0), u(0)) = (x_0, u_0)$*. If for each* $t \geq 0$*,*

$$u(t) = \inf_{\tau > 0} h_{x_0, u_0}(x(t), \tau). \tag{4.1}$$

Then for any t_1*,* $t_2 \in \mathbb{R}_+$ *with* $t_1 \leq t_2$*, there holds*

$$u(t_2) = \inf_{\tau > 0} h_{x(t_1), u(t_1)}(x(t_2), \tau). \tag{4.2}$$

(2) *Given* $(x_0, u_0) \in M \times \mathbb{R}$*, let* $(x(\cdot), u(\cdot)) : \mathbb{R}_- \to M \times \mathbb{R}$ *be a negatively minimizing curve with* $(x(0), u(0)) = (x_0, u_0)$*. If for each* $t \geq 0$*,*

$$u(-t) = \sup_{\tau > 0} h^{x_0, u_0}(x(-t), \tau). \tag{4.3}$$

Then for any t_1*,* $t_2 \in \mathbb{R}_+$ *with* $t_1 \geq t_2$*, there holds*

$$u(-t_1) = \sup_{\tau > 0} h^{x(-t_2), u(-t_2)}(x(-t_1), \tau). \tag{4.4}$$

Proof. We only prove Item (1). Item (2) follows from a similar argument. Since $(x(\cdot), u(\cdot)) : \mathbb{R}_+ \to M \times \mathbb{R}$ is positively minimizing, then

$$u(t_2) \geq \inf_{\tau > 0} h_{x(t_1), u(t_1)}(x(t_2), \tau), \quad \forall\ 0 \leq t_1 < t_2. \tag{4.5}$$

By assumption, for each $t \geq 0$,

$$u(t) = \inf_{\tau > 0} h_{x_0, u_0}(x(t), \tau). \tag{4.6}$$

It follows that

$$u(t_1) = \inf_{\tau > 0} h_{x_0, u_0}(x(t_1), \tau), \quad u(t_2) = \inf_{\tau > 0} h_{x_0, u_0}(x(t_2), \tau),$$

which gives rise to

$$\begin{aligned} u(t_2) &= \inf_{\tau > 0} h_{x_0, u_0}(x(t_2), \tau) \leq \inf_{\tau > 0} h_{x_0, u_0}(x(t_2), t_1 + \tau) \\ &\leq \inf_{\tau > 0} h_{x(t_1), h_{x_0, u_0}(x(t_1), t_1)}(x(t_2), \tau) \\ &= \inf_{\tau > 0} h_{x(t_1), u(t_1)}(x(t_2), \tau). \end{aligned}$$

Combining with (4.5), we get (4.2). □

The following proposition shows that for certain minimizing orbits $(x(\cdot), p(\cdot), u(\cdot)) : \mathbb{R} \to T^*M \times \mathbb{R}$, $p(t)$ is uniquely determined by $(x(t), u(t))$ for all $t \in \mathbb{R}$. Its proof is postponed in Appendix A.1.

Proposition 4.4. *If* $(x, p_0, u) \in \tilde{\mathcal{V}}$*,* $(x, p_+, u) \in \tilde{\mathcal{N}}^+$ *(resp.* $(x, p_-, u) \in \tilde{\mathcal{N}}^-$*), then* $p_0 = p_+$ *(resp.* $p_0 = p_-$*).*

Under (H1)-(H3), by the definitions of $h_{x_0, u_0}(x, t)$ and $h^{x_0, u_0}(x, t)$, we have

Proposition 4.5. *Given* $(x_0, x, t) \in M \times M \times (0, +\infty)$*,* $u, v \in \mathbb{R}$*.*

(1) *for all* u*,* $v \in \mathbb{R}$ *and all* $(x, t) \in M \times (0, +\infty)$*,* $|h_{x_0, u}(x, t) - h_{x_0, v}(x, t)| \leq |u - v|$*;*
(2) *if* $u \geq v$*, then* $h^{x_0, u}(x, t) - h^{x_0, v}(x, t) \geq u - v$*.*

Proof of Lemma 4.2. Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence satisfying

$$\lim_{t_n\to+\infty} h^{x_2,u_2}(x_1,t_n)=u_1.$$

Let $\gamma_n:[0,t_n]\to M$ be a curve $\gamma_n(0)=x_1$ and $\gamma_n(t_n)=x_2$ such that

$$h^{x_2,u_2}(x_1,t_n)=u_2-\int_0^{t_n} L\big(\gamma_n(\tau),\dot\gamma_n(\tau),h^{x_2,u_2}(\gamma_n(\tau),t_n-\tau)\big)d\tau. \tag{4.7}$$

Let $u_n(t):=h^{x_2,u_2}(\gamma_n(t),t_n-t)$. Then $u_n(t_n)=u_2$ and $u_n(0)=h^{x_2,u_2}(x_1,t_n)$. Let

$$p_n(t):=\frac{\partial L}{\partial \dot x}(\gamma_n(t),\dot\gamma_n(t),u_n(t)).$$

We claim that there exists $C>0$ such that

$$\|p_n(0)\|_{x_1}\le C,\quad \|p_n(t_n)\|_{x_2}\le C.$$

In fact, since $u_n(0)\to u_1$ as $n\to+\infty$, combining with the compactness of M, then

$$u_n(1)=h^{x_2,u_2}(\gamma_n(1),t_n-1)=h_{x_1,u_n(0)}(\gamma_n(1),1)$$

is bounded independent of n. Similarly, $u_n(t_n-1)=h^{x_2,u_2}(\gamma_n(t_n-1),1)$ are bounded independent of n. Note that $u_n(t_n)=u_2$. Then one can find $C>0$ such that both $p_n(0)$ and $p_n(t_n)$ is bounded by C (see [20, Appendix] for details).

We assume, up to a subsequence,

$$(\gamma_n(0),p_n(0),u_n(0))\to(x_1,p_1',u_1),\quad (\gamma_n(t_n),p_n(t_n),u_n(t_n))\to(x_2,p_2',u_2).$$

In order to prove Lemma 4.2, it suffices to verify

$$p_1'=p_1,\quad p_2'=p_2.$$

First of all, we prove $p_1'=p_1$. By Proposition 4.4, we only need to show $(x_1,p_1',u_1)\in\tilde{\mathcal{N}}^+$. Let

$$(\bar x(t),\bar p(t),\bar u(t)):=\Phi_t(x_1,p_1',u_1),\quad \forall t\ge 0.$$

By the definition of $\tilde{\mathcal{N}}^+$, we need to show that

(1) the curve $(\bar x(\cdot),\bar u(\cdot)):\mathbb{R}_+\to M\times\mathbb{R}$ is positively minimizing;
(2) for any $t_1,\,t_2\in\mathbb{R}_+$ with $t_1\le t_2$, there holds

$$\bar u(t_2)=\inf_{\tau>0} h_{\bar x(t_1),\bar u(t_1)}(\bar x(t_2),\tau). \tag{4.8}$$

Since $\gamma_n:[0,t_n]\to M$ satisfies (4.7), by the definition of $u_n(t)$, we have

$$u_n(t_1)=h^{\gamma_n(t_2),u_n(t_2)}(\gamma_n(t_1),t_2-t_1),\quad \forall 0\le t_1<t_2\le t_n.$$

By the continuous dependence of solutions to ODEs on the initial data,

$$(\gamma_n(t_1),u_n(t_1))\to(\bar x(t_1),\bar u(t_1)),\quad (\gamma_n(t_2),u_n(t_2))\to(\bar x(t_2),\bar u(t_2)),$$

which combining with the Lipschitz continuity of $(x_0,u_0,x)\mapsto h^{\cdot,\cdot}(\cdot,t_2-t_1)$ yields

$$\bar u(t_1)=h^{\bar x(t_2),\bar u(t_2)}(\bar x(t_1),t_2-t_1),\quad \forall 0\le t_1<t_2<+\infty.$$

By Remark 3.4, $(\bar x(\cdot),\bar u(\cdot)):\mathbb{R}_+\to M\times\mathbb{R}$ is positively minimizing. In order to verify Item (2), by Lemma 4.3(1), we need to prove that for each $t\ge 0$,

$$\bar u(t)=\inf_{\tau>0} h_{x_1,u_1}(\bar x(t),\tau). \tag{4.9}$$

By definition, we have

$$\bar u(t)\ge\inf_{\tau>0} h_{x_1,u_1}(\bar x(t),\tau).$$

It suffices to show

$$\bar{u}(t) \le \inf_{\tau>0} h_{x_1,u_1}(\bar{x}(t),\tau).$$

By contradiction, we assume there exist $t_0, \tau_0 > 0$ such that

$$\bar{u}(t_0) > h_{x_1,u_1}(\bar{x}(t_0),\tau_0).$$

By Proposition 4.5(2), one can find $\delta > 0$ such that

$$h^{\bar{x}(t_0),\bar{u}(t_0)}(x_1,\tau_0) = u_1 + \delta. \tag{4.10}$$

For each $\varepsilon > 0$, there exists n large enough such that

$$d((\gamma_n(t_0), p_n(t_0), u_n(t_0)), (\bar{x}(t_0), \bar{p}(t_0), \bar{u}(t_0))) < \varepsilon,$$

which combining with Lipschitz continuity of $(x_0, u_0) \mapsto h^{\cdot,\cdot}(x_1,\tau_0)$ implies

$$h^{\gamma_n(t_0),u_n(t_0)}(x_1,\tau_0) \ge u_1 + \frac{\delta}{2}.$$

Note that $\gamma_n(t_n) = x_2$, $\gamma_n(0) = x_1$, $u_n(t_n) = u_2$. It follows that

$$\begin{aligned} h^{x_2,u_2}(x_1, t_n - t_0 + \tau_0) &= h^{\gamma_n(t_n),u_n(t_n)}(\gamma_n(0), t_n - t_0 + \tau_0) \\ &\ge h^{\gamma_n(t_0),h^{\gamma_n(t_n),u_n(t_n)}(\gamma_n(t_0),t_n-t_0)}(\gamma_n(0),\tau_0) \\ &= h^{\gamma_n(t_0),u_n(t_0)}(\gamma_n(0),\tau_0) \\ &\ge u_1 + \frac{\delta}{2}. \end{aligned}$$

By assumption, $\limsup_{t\to+\infty} h^{x_2,u_2}(x_1,t) = u_1$. Letting $n \to +\infty$, we have $u_1 > u_1$, which is a contradiction.

Next, we prove $p_2' = p_2$. By Proposition 4.4, we only need to show $(x_2, p_2', u_2) \in \tilde{\mathcal{N}}^-$. Let

$$(\tilde{x}(-t), \tilde{p}(-t), \tilde{u}(-t)) := \Phi_{-t}(x_2, p_2', u_2), \quad \forall 0 \le t < +\infty.$$

We aim to show that

(1) the curve $(\tilde{x}(\cdot), \tilde{u}(\cdot)) : \mathbb{R}_- \to M \times \mathbb{R}$ is negatively minimizing;
(2) for any t_1, $t_2 \in \mathbb{R}_+$ with $t_1 \ge t_2$, there holds

$$\tilde{u}(-t_1) = \sup_{\tau>0} h^{\tilde{x}(-t_2),\tilde{u}(-t_2)}(\tilde{x}(-t_1),\tau). \tag{4.11}$$

The proof of Item (1) is similar to the one of $p_1' = p_1$ above.

Let $\eta_n(-t) := \gamma_n(-t + t_n)$ for each $t \ge 0$. It follows that $\eta_n : [-t_n, 0] \to M$ satisfies (4.7) with $\eta_n(0) = x_2$, $\eta_n(-t_n) = x_1$. Let $v_n(-t) := h^{x_2,u_2}(\eta_n(-t), t)$ for each $t \ge 0$. Then $v_n(0) = u_2$ and

$$v_n(-t_1) = h^{\eta_n(-t_2),v_n(-t_2)}(\eta_n(-t_1), t_1 - t_2), \quad \forall 0 \le t_2 < t_1 < t_n. \tag{4.12}$$

By (4.12),

$$v_n(-t_0) = h_{\eta_n(-t_n),v_n(-t_n)}(\eta_n(-t_0), t_n - t_0). \tag{4.13}$$

Similar to the proof of (4.10), we have

$$h_{\eta_n(-t_0),v_n(-t_0)}(x_2,\tau_0) \le u_2 - \frac{\delta}{2}. \tag{4.14}$$

By Proposition 4.5(1),

$$\begin{aligned}
&|h_{\eta_n(-t_0),h_{x_1,u_1}(\eta_n(-t_0),t_n-t_0)}(x_2,\tau_0)-h_{\eta_n(-t_0),h_{x_1,v_n(-t_n)}(\eta_n(-t_0),t_n-t_0)}(x_2,\tau_0)|\\
\le&|h_{x_1,u_1}(\eta_n(-t_0),t_n-t_0)-h_{x_1,v_n(-t_n)}(\eta_n(-t_0),t_n-t_0)|\\
\le&|u_1-v_n(-t_n))|.
\end{aligned}\tag{4.15}$$

Then we have

$$\begin{aligned}
h_{x_1,u_1}(x_2,t_n-t_0+\tau_0)&\le h_{\eta_n(-t_0),h_{x_1,u_1}(\eta_n(-t_0),t_n-t_0)}(x_2,\tau_0)\\
&\le h_{\eta_n(-t_0),h_{x_1,v_n(-t_n)}(\eta_n(-t_0),t_n-t_0)}(x_2,\tau_0)+|u_1-v_n(-t_n)|\\
&=h_{\eta_n(-t_0),v_n(-t_0)}(x_2,\tau_0)+|u_1-v_n(-t_n)|\\
&\le u_2-\frac{\delta}{2}+|u_1-v_n(-t_n)|,
\end{aligned}$$

where the first inequality is from the Markov property of $h_{x_1,u_1}(x,t)$, the second inequality is owing to (4.15), the first equality is from (4.13), and the last inequality is from (4.14).

Note that $\eta_n(-t_n)=x_1$, $\eta_n(0)=x_2$, $v_n(0)=u_2$. By assumption, $h^{x_2,u_2}(x_1,t_n)\to u_1$ as $t_n\to+\infty$, then $v_n(-t_n)\to u_1$. It is also assumed that $\lim_{t\to+\infty}h_{x_1,u_1}(x_2,t)=u_2$. Letting $n\to+\infty$, we have $u_2<u_2$, which is a contradiction.

This completes the proof of Lemma 4.2. □

5. Topological dynamics around the Aubry set

5.1. **Mather set and recurrence.** For each $v\in\mathcal{S}_-$, $\tilde{\mathcal{N}}_v$ is a flow invariant subset of $T^*M\times\mathbb{R}$. By Proposition 3.9, the set $\tilde{\mathcal{N}}_v$ is non-empty and compact. Then $\tilde{\mathcal{M}}\neq\emptyset$ directly follows from the definition of the Mather set and the assumption (A). Next, we prove

$$\tilde{\mathcal{M}}\subseteq\tilde{\mathcal{N}}\cap\mathrm{cl}(\tilde{\mathcal{R}}).$$

Note that Mather measures are invariant Borel probabilities. Let μ be a Mather measure. By the Poincaré recurrence theorem, one can find a set $A\subseteq\tilde{\mathcal{N}}$ of total μ-measure such that if $(x_0,p_0,u_0)\in A$, then there exist $\{t_m\}_{m\in\mathbb{N}}$ and $\{t_n\}_{n\in\mathbb{N}}$ such that

$$d\left((x_0,p_0,u_0),\Phi_{t_m}(x_0,p_0,u_0)\right)\to0\quad\text{as}\quad t_m\to+\infty,$$

$$d\left((x_0,p_0,u_0),\Phi_{t_n}(x_0,p_0,u_0)\right)\to0\quad\text{as}\quad t_n\to-\infty.$$

Since $\tilde{\mathcal{N}}$ is closed and A is dense in $\mathrm{supp}(\mu)$, then we have

$$\tilde{\mathcal{M}}\subseteq\tilde{\mathcal{N}}\cap\mathrm{cl}(\tilde{\mathcal{R}}).$$

5.2. **Recurrence and strong staticity.** In this part, we aim to show

$$\tilde{\mathcal{N}}\cap\mathrm{cl}(\tilde{\mathcal{R}})\subseteq\tilde{\mathcal{S}}_s.$$

Let $(x(\cdot),u(\cdot)):\mathbb{R}\to M\times\mathbb{R}$ be a semi-static curve. By definition, for each $t_1\le t_2$,

$$u(t_2)=\inf_{s>0}h_{x(t_1),u(t_1)}(x(t_2),s).\tag{5.1}$$

By Proposition 3.3, we have

$$u(t_1)=\sup_{s>0}h^{x(t_2),u(t_2)}(x(t_1),s).\tag{5.2}$$

It remains to prove that for each $t_1>t_2$, (5.2) still holds.

By Proposition 3.8, there exists $v_- \in \mathcal{S}_-$ such that $u(t) = v_-(x(t))$ for all $t \in \mathbb{R}$. If $x(t_1) = x(t_2)$, then

$$u(t_1) = v_-(x(t_1)) = v_-(x(t_2)) = u(t_2),$$

for which (5.2) holds for each $t_1 > t_2$.

In the following, we prove the case with $x(t_1) \neq x(t_2)$. Note that $t_1 > t_2$. Since $(x(\cdot), u(\cdot)) : \mathbb{R} \to M \times \mathbb{R}$ is semi-static, we have

$$u(t_2) = \sup_{s>0} h^{x(t_1),u(t_1)}(x(t_2), s). \tag{5.3}$$

Let $p(t) := \frac{\partial L}{\partial \dot{x}}(x(t), \dot{x}(t), u(t))$. We assume

$$(x(t_1), p(t_1), u(t_1)) \in \tilde{\mathcal{N}} \cap \tilde{\mathcal{R}},$$

Let $\{\tau_n\}_{n\in\mathbb{N}}$ be a sequence such that

$$d\left((x(t_1), p(t_1), u(t_1)), \Phi_{\tau_n}(x(t_1), p(t_1), u(t_1))\right) \to 0 \quad \text{as } \tau_n \to -\infty.$$

Denote $\Delta := t_2 - t_1$. By (5.3), if $\tau_n < \Delta$, we have

$$u(t_1 + \tau_n) = \sup_{s>0} h^{x(t_2),u(t_2)}(x(t_1 + \tau_n), s).$$

Note that $x(t_1) \neq x(t_2)$. It follows from the Lipschitz continuity of Mañé potentials (see Proposition A.2 below) that

$$u(t_1) = \sup_{s>0} h^{x(t_2),u(t_2)}(x(t_1), s),$$

which together with (5.3) implies $(x(\cdot), u(\cdot)) : \mathbb{R} \to M \times \mathbb{R}$ is a strongly static curve. Then $\tilde{\mathcal{N}} \cap \mathrm{cl}(\tilde{\mathcal{R}}) \subseteq \tilde{\mathcal{S}}_s$ follows from the closedness of $\tilde{\mathcal{S}}_s$.

5.3. **Strong staticity and non-wandering property.** In this part, we are devoted to proving

$$\tilde{\mathcal{S}}_s \subseteq \tilde{\mathcal{A}} \cap \tilde{\Omega}.$$

Let $(x(\cdot), u(\cdot)) : \mathbb{R} \to M \times \mathbb{R}$ be a strongly static curve. Let $p(0) := \frac{\partial L}{\partial \dot{x}}(x(0), \dot{x}(0), u(0))$. Let $Q_0 := (x(0), p(0), u(0))$. By definition, it suffices to prove that for each neighborhood U_n of Q_0, there exist $Q_n \in U_n$ and $T_n > 0$ such that $\Phi_{T_n}(Q_n) \in U_n$.

By [18, Theorem 1.4], under (H1)-(H3) and (A), the following function

$$h_{x_0,u_0}(x, +\infty) := \lim_{t\to+\infty} h_{x_0,u_0}(x, t), \quad x \in M \tag{5.4}$$

is well defined. Moreover, for each $s, t \in \mathbb{R}$, both $\lim_{\tau\to+\infty} h_{x(s),u(s)}(x(t), \tau)$. Unfortunately, $\lim_{t\to+\infty} h^{x_0,u_0}(x, t)$ is not always well defined. But it can be proved that $\limsup_{\tau\to+\infty} h^{x(s),u(s)}(x(t), \tau)$ is well defined (see Lemma A.4 below).

Lemma 5.1. *Let $(x(\cdot), u(\cdot)) : \mathbb{R} \to M \times \mathbb{R}$ be a semi-static curve, then*

(1) *it is static if and only if*

$$u(t) = \lim_{\tau\to+\infty} h_{x(s),u(s)}(x(t), \tau), \quad \forall s,\ t \in \mathbb{R};$$

(2) *it is strongly static if and only if*

$$u(t) = \limsup_{\tau\to+\infty} h^{x(s),u(s)}(x(t), \tau), \quad \forall s,\ t \in \mathbb{R}.$$

Proof. We only prove Item (1). Item (2) follows a similar argument. By definition, we have

$$u(t) = \inf_{\tau>0} h_{x(s),u(s)}(x(t),\tau) \leq \lim_{\tau\to+\infty} h_{x(s),u(s)}(x(t),\tau), \quad \forall s,\ t\in\mathbb{R}.$$

On the other hand, for each $n\in\mathbb{N}$, we get

$$u(t) = \inf_{\sigma>0} h_{x(s+n),u(s+n)}(x(t),\sigma).$$

There is a sequence $\{\sigma_n\}\subset\mathbb{R}_+$ such that

$$h_{x(s+n),u(s+n)}(x(t),\sigma_n) < u(t) + \frac{1}{n},$$

which together with the Markov property implies

$$\begin{aligned} h_{x(s),u(s)}(x(t),n+\sigma_n) &\leq h_{x(s+n),h_{x(s),u(s)}(x(s+n),n)}(x(t),\sigma_n) \\ &= h_{x(s+n),u(s+n)}(x(t),\sigma_n) \\ &< u(t) + \frac{1}{n}. \end{aligned}$$

Let $n\to+\infty$. Then

$$u(t) \geq \lim_{\tau\to+\infty} h_{x(s),u(s)}(x(t),\tau).$$

Then we have

$$u(t) = \lim_{\tau\to+\infty} h_{x(s),u(s)}(x(t),\tau), \quad \forall s,\ t\in\mathbb{R}.$$

Conversely, if

$$u(t) = \lim_{\tau\to+\infty} h_{x(s),u(s)}(x(t),\tau), \quad \forall s,\ t\in\mathbb{R},$$

then

$$u(t) \geq \inf_{\tau>0} h_{x(s),u(s)}(x(t),\tau).$$

Note that $(x(\cdot),u(\cdot)) : \mathbb{R}\to M\times\mathbb{R}$ is semi-static. By Proposition 3.8, there exists $v_-\in\mathcal{S}_-$ such that $u(t) = v_-(x(t))$ for each $t\in\mathbb{R}$. Combining Proposition 3.5 and Proposition 3.6, we have

$$\begin{aligned} u(t) = v_-(x(t)) = T^-_\tau v_-(x(t)) &= \inf_{y\in M} h_{y,v_-(y)}(x(t),\tau) \\ &\leq h_{x(s),v_-(x(s))}(x(t),\tau) = h_{x(s),u(s)}(x(t),\tau), \quad \forall \tau>0, \end{aligned}$$

which implies

$$u(t) \leq \inf_{\tau>0} h_{x(s),u(s)}(x(t),\tau).$$

This completes the proof of Lemma 5.1. □

Let $x_0 := x(0)$, $u_0 := u(0)$. By Lemma 5.1,

$$\lim_{t\to+\infty} h_{x_0,u_0}(x_0,t) = u_0,\ \limsup_{t\to+\infty} h^{x_0,u_0}(x_0,t) = u_0.$$

Then $\tilde{\mathcal{S}}_s \subseteq \tilde{\mathcal{A}}\cap\tilde{\Omega}$ follows from Lemma 4.1.

Remark 5.2. *The Aubry set in the classical case is defined in T^*M instead of $T^*M\times\mathbb{R}$. One can embed this set into $T^*M\times\mathbb{R}$ by adding the u-argument in the following way. An important result in weak KAM theory ([8, Theorem 5.2.8]) shows that there exists a conjugate pair (u_-,u_+) such that the Aubry set is represented as*

$$\tilde{\mathcal{I}}_{(u_-,u_+)} := \{(x,p)\in T^*M \mid u_-(x) = u_+(x),\ p = Du_-(x)\}.$$

The embedding Aubry set in $T^*M \times \mathbb{R}$ is

$$\tilde{\mathcal{A}} := \{(x,p,u) \in T^*M \times \mathbb{R} \mid (x,p) \in \tilde{\mathcal{I}}_{(u_-,u_+)},\ u = u_-(x).\}$$

Note that $\tilde{\mathcal{A}} = \tilde{\mathcal{S}}_s$ in the classical case. From Theorem 1, each embedding Aubry set is non-wandering in $T^*M \times \mathbb{R}$ in classical cases. Moreover, each non-wandering set in $T^*M \times \mathbb{R}$ is also an embedding Aubry set by choosing certain conjugate pair (see [10, Theorem 1.5]). This gives a description for the Aubry set in the classical case without using action minimizing property.

6. Strictly increasing case

In this section, we consider the cases under (H1), (H2), (H3') and (A).

6.1. **The structure of $\mathcal{S}_+$.** It was shown by [22, Proposition 12] that

Proposition 6.1. *All of elements in $\mathcal{S}_+$ are uniformly bounded and equi-Lipschitz continuous.*

Note that $(\mathcal{S}_+, \preceq)$ is a partially ordered set. In view of Zorn's lemma, if every chain in $\mathcal{S}_+$ has a lower bound in $\mathcal{S}_+$, then $\mathcal{S}_+$ contains a minimal element. To prove Item (1) of Theorem 2, it is suffices to show

Proposition 6.2. *Let $\mathcal{Z}$ be a totally ordered subset of $\mathcal{S}_+$. Let $\check{u}(x) := \inf_{u\in\mathcal{Z}} u(x)$ for each $x \in M$. Then $\check{u} \in \mathcal{S}_+$.*

Lemma 6.3. *There exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{Z}$ such that u_n converges to $\check{u}$ uniformly.*

Proof. Note that all of elements in $\mathcal{S}_+$ are uniformly bounded and κ-equi-Lipschitz continuous. We only need to construct a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{Z}$ such that u_n converges to $\check{u}$ pointwisely.

Since the Riemannian manifold M is compact, it is separable. Namely on can find a countable dense subset denoted by $U := \{x_1, x_2, \dots, x_n, \dots\}$.

Claim. There exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{Z}$ such that for a given $n \in \mathbb{N}$ and each $i \in \{1, 2, \dots, n\}$,

$$0 \le u_n(x_i) - \check{u}(x_i) < \frac{1}{n}. \tag{6.1}$$

If the claim is true, then u_n converges to $\check{u}$ pointwisely. In fact, according to Proposition 6.1, every $u_+ \in \mathcal{S}_+$ is κ-Lipschitz, we have

$$\check{u}(x) - \check{u}(y) \le \sup_{u_+\in\mathcal{S}_+} |u_+(x) - u_+(y)| \le \kappa \mathrm{dist}(x,y). \tag{6.2}$$

Fix $x \in M$. There exists a subsequence $V := \{x_{k_m}\}_{m\in\mathbb{N}} \subseteq U$ such that $|x_{k_m} - x| < 1/k_m$. Given $k_m \in \mathbb{N}$, we take $n \ge k_m$. Then $\{x_1, x_2, \dots, x_n\} \cap V \neq \emptyset$. Let $x_{i_0} \in \{x_1, x_2, \dots, x_n\} \cap V$. It follows from (6.2) that

$$\begin{aligned}|u_n(x) - \check{u}(x)| &\le |u_n(x) - u_n(x_{i_0})| + |u_n(x_{i_0}) - \check{u}(x_{i_0})| + |\check{u}(x) - \check{u}(x_{i_0})| \\ &\le 2\kappa \ \mathrm{dist}(x_{i_0}, x) + \frac{1}{n} \le \frac{2\kappa}{i_0} + \frac{1}{n}.\end{aligned}$$

Let n and i_0 tend to $+\infty$ successively. We get the pointwise convergence of u_n to $\check{u}$.

We prove the claim by induction in the following. For $x_1 \in U$, by the definition of the infimum, there exists $u_1 \in \mathcal{Z}$ such that $u_1(x_1) - \check{u}(x_1) < 1/n$ for a given $n \in \mathbb{N}$.

We assume there exists $u_k \in \mathcal{Z}$ such that for $i \in \{1, 2, \dots, k\}$, $u_k(x_i) - \breve{u}(x_i) < 1/n$. One needs to construct u_{k+1}. For $x_{k+1} \in U$, if $u_k(x_{k+1}) - \breve{u}(x_{k+1}) < 1/n$, then we take $u_{k+1} \equiv u_k$. Otherwise, we have $u_k(x_{k+1}) - \breve{u}(x_{k+1}) \geq 1/n$. In this case, one can find $u_{k+1} \in \mathcal{Z}$ such that $u_{k+1}(x_{k+1}) - \breve{u}(x_{k+1}) < 1/n$.

It remains to show $u_{k+1}(x_i) - \breve{u}(x_i) < 1/n$ for $i \in \{1, 2, \dots, k\}$. We know that

$$u_{k+1}(x_{k+1}) < \breve{u}(x_{k+1}) + \frac{1}{n} \leq u_k(x_{k+1}).$$

Note that $\mathcal{Z}$ is totally ordered. It yields $u_{k+1} \leq u_k$ on M. Thus, for $i \in \{1, 2, \dots, k\}$,

$$u_{k+1}(x_i) - \breve{u}(x_i) \leq u_k(x_i) - \breve{u}(x_i) < \frac{1}{n}.$$

This completes the proof of Lemma 6.3. □

Under (H1)-(H3), by the definitions of $T_t^\pm$, we have (see [21, Proposition 2.4] for details)

Proposition 6.4.

(1) *For φ_1 and $\varphi_2 \in C(M)$, if $\varphi_1(x) < \varphi_2(x)$ for all $x \in M$, we have $T_t^- \varphi_1(x) < T_t^- \varphi_2(x)$ and $T_t^+ \varphi_1(x) < T_t^+ \varphi_2(x)$ for all $(x, t) \in M \times (0, +\infty)$.*

(2) *Given any φ and $\psi \in C(M)$, we have $\|T_t^- \varphi - T_t^- \psi\|_\infty \leq \|\varphi - \psi\|_\infty$ and $\|T_t^+ \varphi - T_t^+ \psi\|_\infty \leq e^{\lambda t} \|\varphi - \psi\|_\infty$ for all $t > 0$.*

Proof of Proposition 6.2. If $\mathcal{Z}$ is a finite set, the proof is finished. We then consider $\mathcal{Z}$ being a infinite set. By Proposition 3.6, we need to show $\breve{u}$ is a fixed point of T_t^+. By Proposition 6.4 (2), we have

$$\|T_t^+ u_n - T_t^+ \breve{u}\|_\infty \leq e^{\lambda t} \|u_n - \breve{u}\|_\infty,$$

By Lemma 6.3, the right hand side tends to zero. Then for a given $t > 0$,

$$T_t^+ \breve{u} = \lim_{n \to +\infty} T_t^+ u_n = \lim_{n \to +\infty} u_n = \breve{u},$$

which implies $\breve{u} \in \mathcal{S}_+$ by Proposition 3.6. □

Next, we prove Item (2) of Theorem 2. Let us recall $\mathcal{Z}_{\max}$ denotes a maximal totally ordered subset of $\mathcal{S}_+$, u_- denotes the unique backward weak KAM solution, and u^* denotes the minimal element in $\mathcal{Z}_{\max}$, i.e.

$$u^*(x) := \inf_{v_+ \in \mathcal{Z}_{\max}} v_+(x).$$

Lemma 6.5. *Denote*

$$\mathcal{I}_{u^*} := \{x \in M \mid u^*(x) = u_-(x)\}.$$

Then $\mathcal{I}_{u^*} \neq \emptyset$ and for all $v_+ \in \mathcal{Z}_{\max}$, $v_+ = u_-$ on $\mathcal{I}_{u^*}$.

Proof. By Proposition 6.2, $u^* \in \mathcal{S}_+$. By Proposition 3.7 and the uniqueness of u_-,

$$u_-(x) = \lim_{t \to +\infty} T_t^- u^*(x).$$

According to Proposition 3.9, $\mathcal{I}_{u^*} \neq \emptyset$. Note that for all $v_+ \in \mathcal{Z}_{\max}$,

$$u^* \leq v_+ \leq u_-, \quad \text{on } M.$$

Then $v_+(x) = u_-(x)$ for all $x \in \mathcal{I}_{u^*}$. □

Lemma 6.6. *Define $m(\cdot,\cdot) : M \times M \to \mathbb{R}$ by*

$$m(x,y) := \limsup_{t\to+\infty} h^{y,u_-(y)}(x,t), \quad \forall x,y \in M.$$

Then for each $y \in M$, $m(\cdot,y) \in \mathcal{S}_+$.

Proof. Fix $y \in M$. By [22, Lemma 1(i)], $h^{y,u_-(y)}(\cdot,\cdot)$ is uniformly bounded on $M \times [\delta,+\infty)$ for any $\delta > 0$. Thus, there exists a constant $K > 0$ independent of t such that for $t > \delta$ and each $x \in M$,

$$|h^{y,u_-(y)}(x,t)| \le K.$$

Note that for any $t > 2\delta$, we have

$$\begin{aligned}
&\left|h^{y,u_-(y)}(x,t) - h^{y,u_-(y)}(x',t)\right| \\
=&\left|\sup_{z\in M} h^{z,h^{y,u_-(y)}(z,t-\delta)}(x,\delta) - \sup_{z\in M} h^{z,h^{y,u_-(y)}(z,t-\delta)}(x',\delta)\right| \\
\le& \sup_{z\in M}\left|h^{z,h^{y,u_-(y)}(z,t-\delta)}(x,\delta) - h^{z,h^{y,u_-(y)}(z,t-\delta)}(x',\delta)\right|.
\end{aligned}$$

Since $h^{\cdot,\cdot}(\cdot,\delta)$ is uniformly Lipschitz on $M \times [-K,K] \times M$ with some Lipschitz constant denoted by ι, then

$$\left|h^{y,u_-(y)}(x,t) - h^{y,u_-(y)}(x',t)\right| \le \iota \operatorname{dist}(x,x'), \quad \forall t > 2\delta.$$

It follows that the family $\{h^{y,u_-(y)}(x,t)\}_{t>2\delta}$ is equi-Lipschitz continuous with respect to x. Thus, $m(x,y)$ is well defined. Note that for a given $t > 0$, the Lax-Oleinik semigroup T_t^+ satisfies

$$\|T_t^+\varphi - T_t^+\psi\|_\infty \le e^{\lambda t}\|\varphi - \psi\|_\infty,$$

for any $\varphi, \psi \in C(M,\mathbb{R})$. Note that T_t^+ commutes with $\limsup$. It follows that for a given $t \ge 0$,

$$T_t^+ m(x,y) = \limsup_{s\to+\infty} T_t^+ h^{y,u_-(y)}(x,s) = \limsup_{s\to+\infty} h^{y,u_-(y)}(x,s+t) = m(x,y),$$

which implies $m(\cdot,y) \in \mathcal{S}_+$. □

Let us recall $\mathcal{V} = \pi^*\tilde{\mathcal{V}}$, where $\tilde{\mathcal{V}}$ denotes the set of $(x,p,u) \in T^*M \times \mathbb{R}$, for which there exists a strongly static orbit

$$(x(\cdot),p(\cdot),u(\cdot)) : \mathbb{R} \to T^*M \times \mathbb{R}$$

passing through (x,p,u).

Lemma 6.7. *Given $y \in \mathcal{V}$, we have*

$$\inf_{\substack{v_+(y)=u_-(y)\\ v_+\in\mathcal{S}_+}} v_+(x) = m(x,y), \quad \forall x \in M. \tag{6.3}$$

Proof. By the definition of $\mathcal{V}$, there exists a strongly static curve

$$(x(\cdot),u(\cdot)) : \mathbb{R} \to M \times \mathbb{R}$$

such that $x(0) = y$. Since $\mathcal{V} \subseteq \mathcal{A}$, then $u(t) = u_-(x(t))$ for all $t \in \mathbb{R}$. In particular, $u(0) = u_-(y)$. It follows from Proposition 5.1 that

$$u_-(y) = u(0) = \limsup_{s\to+\infty} h^{x(0),u(0)}(x(0),s) = \limsup_{s\to+\infty} h^{y,u_-(y)}(y,s) = m(y,y),$$

which together with $m(\cdot,y)\in\mathcal{S}_+$ implies

$$m(x,y)\geq \inf_{\substack{v_+(y)=u_-(y)\\ v_+\in\mathcal{S}_+}} v_+(x).$$

On the other hand, we have

$$\begin{aligned} v_+(x)=T_t^+v_+(x) &\geq \sup_{t>0} h^{y,v_+(y)}(x,t)\\ &=\sup_{t>0} h^{y,u_-(y)}(x,t)\geq \limsup_{t\to+\infty} h^{y,u_-(y)}(x,t)=m(x,y). \end{aligned}$$

It means

$$\inf_{\substack{v_+(y)=u_-(y)\\ v_+\in\mathcal{S}_+}} v_+(x)\geq m(x,y).$$

This completes the proof of Lemma 6.7. □

Remark 6.8. *In general contact cases,*

(1) *$m(x,y)$ is Lipschitz continuous in x, but it may not be continuous in y;*
(2) *the equality (6.3) may not hold for $y\in M\backslash\mathcal{V}$.*

We still consider the Hamilton-Jacobi equation in Proposition 2.11:

$$\lambda u+\frac{1}{2}|Du|^2+Du\cdot V(x)=0,\qquad x\in\mathbb{T}, \tag{6.4}$$

where $0<\lambda<|V'(x_2)|$. By definition, for $y_0\in\mathbb{T}$,

$$m(y_0,y_0)=-\liminf_{\tau\to+\infty}\inf_{\gamma(0)=\gamma(\tau)=y_0}\int_0^\tau e^{\lambda s}\frac{1}{2}|\dot\gamma(s)-V(\gamma(s))|^2ds.$$

By Lemma 6.6, $m(\cdot,y_0)\in\mathcal{S}_+$. We choose a point $y_0\neq x_1,x_2$. It is not difficult to verify $m(y_0,y_0)<0$. It follows that $m(x,y_0)=w_+(x)$ for all $x\in\mathbb{T}$. On the other hand, by Lemma 5.1(2), $m(x_2,x_2)=0$. Thus, $m(x,x_2)=u_+(x)\equiv 0$ for all $x\in\mathbb{T}$. Then

$$\lim_{\substack{y\to x_2\\ y\neq x_2}} m(y_0,y)=w_+(y_0)<0=m(y_0,x_2),$$

which means $m(x,y)$ is not continuous at $y=x_2$. More precisely, for each $x\in\mathbb{T}$, $m(x,y)$ is continuous at $y\neq x_2$ and it is upper semicontinuous at $y=x_2$. This verifies Item (1).

For Item (2), we already know that if $y_0\neq x_1,x_2$, then $m(x,y_0)=w_+(x)$ for all $x\in\mathbb{T}$. Then

$$\inf_{\substack{v_+(y_0)=u_-(y_0)\\ v_+\in\mathcal{S}_+}} v_+(y_0)=u_+(y_0)=0>w_+(y_0)=m(y_0,y_0).$$

Thus, the equality (6.3) does not hold.

Proof of Theorem 2(2). By the definition of the Mather set, we have

$$\emptyset\neq\tilde{\mathcal{M}}_{u^*}:=\tilde{\mathcal{M}}\cap G_{u^*}\subseteq\tilde{\mathcal{I}}_{u^*}.$$

Based on Section 5.1, the recurrent points are dense in $\tilde{\mathcal{M}}_{u^*}$. Let $(x_0,p_0,u_0)\in\tilde{\mathcal{M}}_{u^*}$ be a recurrent point. According to Section 5.2, $(x_0,p_0,u_0)\in\tilde{\mathcal{V}}$. Then one can choose

$$x_0\in\mathcal{M}\cap\mathcal{V}\cap\mathcal{I}_{u^*}, \tag{6.5}$$

such that

$$\inf_{\substack{v_+(x_0)=u_-(x_0)\\ v_+\in\mathcal{S}_+}} v_+(x)=m(x,x_0),\quad \forall x\in M.$$

It remains to prove

$$\inf_{\mathcal{Z}_{\max}} v_+(x) = \inf_{\substack{v_+(x_0)=u_-(x_0)\\ v_+\in\mathcal{S}_+}} v_+(x).$$

By (6.5), $u^*(x_0) = u_-(x_0)$. It follows that for all $v_+ \in \mathcal{Z}_{\max}$, $v_+(x_0) = u_-(x_0)$. Then

$$\inf_{\mathcal{Z}_{\max}} v_+(x) \geq \inf_{\substack{v_+(x_0)=u_-(x_0)\\ v_+\in\mathcal{S}_+}} v_+(x).$$

On the other hand, by Lemma 6.6 and Lemma 6.7,

$$u(x) := \inf_{\substack{v_+(x_0)=u_-(x_0)\\ v_+\in\mathcal{S}_+}} v_+(x) \in \mathcal{S}_+.$$

Due to the maximality of the set $\mathcal{Z}_{\max}$, we have $u \in \mathcal{Z}_{\max}$. It implies

$$\inf_{\mathcal{Z}_{\max}} v_+(x) \leq \inf_{\substack{v_+(x_0)=u_-(x_0)\\ v_+\in\mathcal{S}_+}} v_+(x).$$

This completes the proof of Theorem 2(2). □

6.2. **Existence of transitive orbits.** We prove Theorem 3 in this part. By Lemma 4.2, we only need to consider the case with $x_2 \in \mathcal{V}$. By Proposition 6.6 and Proposition 6.7, the function

$$m(\cdot, x_2) = \limsup_{t\to+\infty} h^{x_2,u_-(x_2)}(\cdot, t)$$

is the minimal forward weak KAM solution of (HJ) equaling to $u_-(x_2)$ at x_2. By assumption, for each $v_+ \in \mathcal{S}_+$, $v_+(x_2) = u_-(x_2)$ implies $v_+(x_1) = u_-(x_1)$. Then

$$\limsup_{t\to+\infty} h^{x_2,u_-(x_2)}(x_1, t) = u_-(x_1). \tag{6.6}$$

Note that for each $(x_0, u_0) \in M \times \mathbb{R}$,

$$\lim_{t\to+\infty} h_{x_0,u_0}(x, t) = u_-(x).$$

It yields

$$\lim_{t\to+\infty} h_{x_1,u_1}(x_2, t) = u_-(x_2)$$

Note that $u_-(x_1) = u_1$, $u_-(x_2) = u_2$. Theorem 3 follows from Lemma 4.1. □

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Appendix A. Auxiliary results

For the sake of generality, we will prove all of the results in this appendix under (H1), (H2) and $|\frac{\partial H}{\partial u}| \leq \lambda$ instead of (H3).

A.1. **Strong staticity and one-sided semi-staticity.** In this part, we prove Proposition 4.4. First of all, we provide a way to construct "long" minimizers from "short" ones, which is a direct consequence of the Markov and monotonicity properties of the action functions.

Proposition A.1. *Given any x, y and $z \in M$, u_1, u_2 and $u_3 \in \mathbb{R}$, t, $s > 0$, let*

$$h_{x,u_1}(y,t) = u_2, \quad h_{y,u_2}(z,s) = h_{x,u_1}(z,t+s) = u_3$$

$$(\textit{resp.} \quad h^{z,u_3}(y,s) = u_2, \quad h^{y,u_2}(x,t) = h^{z,u_3}(x,t+s) = u_1).$$

Let $\gamma_1 : [0,t] \to M$ be a minimizer of $h_{x,u_1}(y,t)$ (resp. $h^{y,u_2}(x,t)$) and $\gamma_2 : [0,s] \to M$ be a minimizer of $h_{y,u_2}(z,s)$ (resp. $h^{z,u_3}(y,s)$). Then

$$\gamma(\sigma) := \begin{cases} \gamma_1(\sigma), & \sigma \in [0,t], \\ \gamma_2(\sigma - t), & \sigma \in [t, t+s], \end{cases}$$

is a minimizer of $h_{x,u_1}(z,t+s)$ (resp. $h^{z,u_3}(x,t+s)$).

Proof of Proposition 4.4. We only need to prove if $(x,p_0,u) \in \tilde{\mathcal{V}}$, $(x,p_+,u) \in \tilde{\mathcal{N}}^+$, then $p_0 = p_+$. The other case is similar. For each $t \in \mathbb{R}$, let

$$(x_1(t), p_1(t), u_1(t)) = \Phi_t(x, p_0, u).$$

For each $t \geq 0$, let

$$(x_2(t), p_2(t), u_2(t)) = \Phi_t(x, p_+, u).$$

We need to prove if a globally minimizing curve $(x_1(\cdot), u_1(\cdot)) : \mathbb{R} \to M \times \mathbb{R}$ satisfies for each $t_1, t_2 \in \mathbb{R}$,

$$u_1(t_2) = \inf_{s>0} h_{x_1(t_1),u_1(t_1)}(x_1(t_2), s), \tag{A.1}$$

then $p_0 = p_+$.

Since $(x,p_+,u) \in \tilde{\mathcal{N}}^+$, then $(x_2(\cdot), u_2(\cdot)) : \mathbb{R}_+ \to M \times \mathbb{R}$ is positively semi-static. Fixing $\delta > 0$, by the Markov property, we have

$$h_{x_1(-\delta),u_1(-\delta)}(x_2(\delta), 2\delta) = \inf_{y \in M} h_{y,h_{x_1(-\delta),u_1(-\delta)}(y,\delta)}(x_2(\delta), \delta).$$

Note that

$$h_{x_1(-\delta),u_1(-\delta)}(x, \delta) = u, \quad h_{x,u}(x_2(\delta), \delta) = u_2(\delta).$$

It follows that

$$h_{x_1(-\delta),u_1(-\delta)}(x_2(\delta), 2\delta) \leq h_{x,u}(x_2(\delta), \delta).$$

We assert that the inequality above is indeed an equality. If the assertion is true, then by Proposition A.1, the curve defined by

$$\gamma(\sigma) := \begin{cases} x_1(\sigma - \delta), & \sigma \in [0, \delta], \\ x_2(\sigma - \delta), & \sigma \in [\delta, 2\delta], \end{cases}$$

is a minimizer of $h_{x_1(-\delta),u_1(-\delta)}(x_2(\delta), 2\delta)$ and it is of class C^1. Thus,

$$p_0 = \frac{\partial L}{\partial \dot{x}}(x, \dot{\gamma}(0), 0) = p_+.$$

It remains to verify the assertion. By contradiction, we assume that there exists $\Delta > 0$ such that

$$h_{x_1(-\delta),u_1(-\delta)}(x_2(\delta), 2\delta) = h_{x,u}(x_2(\delta), \delta) - \Delta.$$

By (A.1), for each $\varepsilon > 0$, one can find $s_0 > 0$ such that

$$|h_{x,u}(x_1(-\delta), s_0) - u_1(-\delta)| \leq \varepsilon.$$

From the Lipschitz continuity of $h_{x_0,u_0}(x,t)$ w.r.t. u_0,

$$|h_{x_1(-\delta),h_{x,u}(x_1(-\delta),s_0)}(x_2(\delta),2\delta) - h_{x_1(-\delta),u_1(-\delta)}(x_2(\delta),2\delta)| \leq k\varepsilon,$$

where k denotes the Lipschitz constant of $h_{x_0,u_0}(x,t)$ w.r.t. u_0. It follows from the definition of $\tilde{\mathcal{N}}^+$ that

$$\begin{aligned} u_2(\delta) &= \inf_{\tau>0} h_{x,u}(x_2(\delta),\tau), \\ &\leq h_{x,u}(x_2(\delta), s_0+2\delta), \\ &\leq h_{x_1(-\delta),h_{x,u}(x_1(-\delta),s_0)}(x_2(\delta),2\delta), \\ &\leq h_{x_1(-\delta),u_1(-\delta)}(x_2(\delta),2\delta) + k\varepsilon, \\ &= h_{x,u}(x_2(\delta),\delta) - \Delta + k\varepsilon. \end{aligned}$$

Note that Δ, k are constants independent of ε. Taking ε small enough, we have

$$u_2(\delta) \leq h_{x,u}(x_2(\delta),\delta) - \frac{\Delta}{2} = u_2(\delta) - \frac{\Delta}{2},$$

which is a contradiction. This completes the proof of Proposition 4.4.

A.2. **Lipschitz continuity of Mañé potentials.** Let $(x(\cdot),u(\cdot)) : \mathbb{R} \to M \times \mathbb{R}$ be a semi-static curve. Fixing $\tau \in \mathbb{R}$, we consider two kinds of the Mañé potentials as follows:

$$\check{K}_\tau(x) := \inf_{s>0} h_{x(\tau),u(\tau)}(x,s), \quad \hat{K}_\tau(x) := \sup_{s>0} h^{x(\tau),u(\tau)}(x,s).$$

In this part, we will prove

Proposition A.2. *Given $\tau \in \mathbb{R}$, let U be an open set containing $x(\tau)$. Then both $\check{K}_\tau(x)$ and $\hat{K}_\tau(x)$ are uniformly Lipschitz continuous with respect to $x \in M\backslash U$.*

We only need to prove this proposition for $\check{K}_\tau(x)$, from which the Lipschitz continuity of $\hat{K}_\tau(x)$ can be obtained by a similar way.

Lemma A.3. *Let $\mathcal{K}$ be a compact subset of M and $u_0 \in \mathbb{R}$. Then for any $x_0 \in M\backslash\mathcal{K}$, we have $\lim_{t\to 0^+} h_{x_0,u_0}(x,t) = +\infty$ uniformly in $x \in \mathcal{K}$.*

Proof. Given $(x,t) \in \mathcal{K} \times (0,+\infty)$, let $\Gamma^t_{x_0,x}$ be the set of the minimizers of $h_{x_0,u_0}(x,t)$. Namely, for each $\gamma : [0,t] \to M$ contained in $\Gamma^t_{x_0,x}$, we have $\gamma(0) = x_0$, $\gamma(t) = x$ and

$$h_{x_0,u_0}(x,t) = u_0 + \int_0^t L\big(\gamma(\tau),\dot\gamma(\tau),h_{x_0,u_0}(\gamma(\tau),\tau)\big)d\tau. \tag{A.2}$$

Let

$$g_x(t) := \inf_{\gamma\in\Gamma^t_{x_0,x}} \sup_{0\leq s\leq t} h_{x_0,u_0}(\gamma(s),s).$$

We proceed the remaining proof by two steps.

Step 1: we show $\lim_{t\to 0^+} g_x(t) = +\infty$, uniformly in $x \in \mathcal{K}$.

By contradiction, we assume there exist $x_n \in \mathcal{K}$ and $\gamma_n \in \Gamma^{t_n}_{x_0,x_n}$ with $t_n \to 0$ as $n \to +\infty$ such that

$$h_{x_0,u_0}(\gamma_n(s),s) < C_1, \quad \forall s \in [0,t_n], \tag{A.3}$$

where C_1 is a constant independent of n. Let $A := \inf_{(x,\dot x)\in TM} L(x,\dot x,u_0)$. Given $T_0 > 0$, let

$$C_2 := |u_0|e^{\lambda T_0} + \frac{|A-\lambda u_0|}{\lambda}\left(e^{\lambda T_0}-1\right) + 1,$$

where λ is a Lipschitz constant of $H(x,p,u)$ w.r.t. u.

Claim. For any $(x,t)\in M\times(0,T_0]$, $h_{x_0,u_0}(x,t)>-C_2$.

Proof of the claim. We assume by contradiction that there exists $(x_1,t_1)\in M\times(0,T_0]$ such that $h_{x_0,u_0}(x_1,t_1)\leq -C_2$. Let $\gamma\in\Gamma^{t_1}_{x_0,x_1}$. Denote $u(s):=h_{x_0,u_0}(\gamma(s),s)$ for $s\in[0,t_1]$. Note that $u_0\geq -C_2$. Since $u(s)$ is continuous on $(0,t_1]$, and $\lim_{s\to 0^+}u(s)=u_0$, there exists a closed interval $[s_1,s_2]\subseteq[0,t_1]$ such that

$$u(s_1)=u_0,\quad u(s_2)=-C_2,\quad -C_2\leq u(s)\leq u_0,\quad \forall s\in[s_1,s_2].$$

Since γ satisfies (A.2), based on the variational principle (see Proposition 2.2),

$$\dot u(s)=L(\gamma(s),\dot\gamma(s),u(s))\geq A+\lambda(u(s)-u_0),\ \ \forall s\in[s_1,s_2].$$

A direct calculation yields for any $s\in[s_1,s_2]$,

$$\begin{aligned}u(s)&\geq u_0e^{\lambda(s-s_1)}+\frac{A-\lambda u_0}{\lambda}\left(e^{\lambda(s-s_1)}-1\right)\\&\geq -|u_0|e^{\lambda T_0}-\frac{|A-\lambda u_0|}{\lambda}\left(e^{\lambda T_0}-1\right)\\&>-C_2.\end{aligned}$$

This contradicts $u(s_2)=-C_2$. Then the claim is true.

For n large enough, we have $t_n<T_0$. Let $C:=\max\{C_1,C_2\}$. Based on (A.3) and the assertion above,

$$|h_{x_0,u_0}(\gamma_n(s),s)|\leq C,\quad \forall s\in[0,t_n]. \tag{A.4}$$

Let $\delta:=\mathrm{dist}(x_0,\mathcal{K})$, where $\mathrm{dist}(\cdot,\cdot)$ denotes a distance induced by the Riemannian metric on M. Let

$$B:=\frac{C+1+|u_0|}{\delta}.$$

Since $L(x,\dot x,0)$ is superlinear in $\dot x$, then there is $D:=D(B)\in\mathbb{R}$ such that $L(x,\dot x,0)\geq B\|\dot x\|_x-D$ for all $(x,\dot x)\in TM$. Since $t_n\to 0^+$ as $n\to+\infty$, for n large enough, we get $|(D+\lambda C)t_n|<1$. Note that

$$\begin{aligned}h_{x_0,u_0}(x_n,t_n)&=u_0+\int_0^{t_n}L(\gamma_n(s),\dot\gamma_n(s),h_{x_0,u_0}(\gamma_n(s),s))ds\\&\geq u_0+\int_0^{t_n}L(\gamma_n(s),\dot\gamma_n(s),0)ds-\lambda\int_0^{t_n}|h_{x_0,u_0}(\gamma_n(s),s)|ds\\&\geq u_0+B\delta-Dt_n-\lambda Ct_n\\&=u_0+B\delta-(D+\lambda C)t_n\\&>C,\end{aligned}$$

which contradicts (A.4). Therefore, $\lim_{t\to 0^+}g_x(t)=+\infty$, uniformly in $x\in\mathcal{K}$.

Step 2: we show $\lim_{t\to 0^+}h_{x_0,u_0}(x,t)=+\infty$, uniformly for all $x\in\mathcal{K}$.

From Step 1, for any $N>0$, there is $t_N>0$ such that $g_x(t)>N$ for $t<t_N$ and all $x\in\mathcal{K}$. Let $\gamma\in\Gamma^t_{x_0,x}$. Note that $h_{x_0,u_0}(\gamma(s),s)\to u_0$ as $s\to 0^+$. One can find $s_0\in[0,t]$ such that $h_{x_0,u_0}(\gamma(s_0),s_0)=N$.

Note that

$$h_{x_0,u_0}(x,t)=h_{x_0,u_0}(\gamma(s_0),s_0)+\int_{s_0}^t L(\gamma(s),\dot\gamma(s),h_{x_0,u_0}(\gamma(s),s))ds.$$

Similar to the argument above, for $t < t_N$, we have

$$\begin{aligned} h_{x_0,u_0}(x,t) &\geq N + \int_{s_0}^t L(\gamma(s),\dot\gamma(s),0)ds - \lambda \int_{s_0}^t |h_{x_0,u_0}(\gamma(s),s)|ds \\ &\geq N + B\delta - (D+\lambda C)(t-s_0), \end{aligned}$$

Let $t \to 0^+$. Then $t - s_0 \to 0^+$. Moreover, $h_{x_0,u_0}(x,t) > N$ for each $x \in \mathcal{K}$, which completes the proof. ☐

By [23, Lemma C.1], we have

Lemma A.4. *Let $(x(\cdot),u(\cdot)) : \mathbb{R} \to M \times \mathbb{R}$ be a semi-static curve. Then for each $\delta > 0$,*

- *Uniform Boundedness: there exists a constant $K > 0$ independent of t such that for $t > \delta$ and each $x \in M$, $s \in \mathbb{R}$,*
$$|h_{x(s),u(s)}(x,t)| \leq K, \quad |h^{x(s),u(s)}(x,t)| \leq K;$$
- *Equi-Lipschitz Continuity: there exists a constant $\kappa > 0$ independent of t such that for $t > 2\delta$ and $s \in \mathbb{R}$, both $x \mapsto h_{x(s),u(s)}(x,t)$ and $x \mapsto h^{x(s),u(s)}(x,t)$ are κ-Lipschitz continuous on M.*

Proof of Proposition A.2. We only need to prove this proposition for $\check{K}_\tau(x)$. Let U be an open neighborhood of $x(\tau)$ and $x \in \mathcal{K} := M\backslash U$. By Lemma A.3, we have $\lim_{t\to 0^+} h_{x(\tau),u(\tau)}(x,t) = +\infty$ uniformly for $x \in \mathcal{K}$. Thus, there exists $\delta > 0$ independent of $x \in \mathcal{K}$ such that

$$\check{K}_\tau(x) := \inf_{s>0} h_{x(\tau),u(\tau)}(x,s) = \inf_{s>\delta} h_{x(\tau),u(\tau)}(x,s), \quad \forall x \in \mathcal{K}.$$

It follows from Lemma A.4 that

$$\begin{aligned} &\left|\check{K}_\tau(x) - \check{K}_\tau(y)\right| \\ =&\left|\inf_{s>\delta} h_{x(\tau),u(\tau)}(x,s) - \inf_{s>\delta} h_{x(\tau),u(\tau)}(y,s)\right| \\ \leq& \sup_{s>\delta}\left|h_{x(\tau),u(\tau)}(x,s) - h_{x(\tau),u(\tau)}(y,s)\right| \\ \leq& \kappa\, d(x,y). \end{aligned}$$

This completes the proof. ☐

Appendix B. Proof of Proposition 2.11

It is clear that $u_- \equiv 0$ is the unique element in $\mathcal{S}_-$, and $u_+ \equiv 0 \in \mathcal{S}_+$.

B.1. **On Item (i).** The contact Hamilton equation reads

$$\begin{cases} \dot x = p + V(x), \\ \dot p = -pV'(x) - \lambda p, \\ \dot u = p(p+V(x)) - H(x,p,u). \end{cases} \tag{B.1}$$

Denote the solution of (B.1) by $(x(t),p(t),u(t))$.

A direct calculation shows

$$\frac{dH}{dt} = -\lambda H(x(t),p(t),u(t)).$$

Thus, we only need to consider the dynamics on zero energy level set

$$E := \{(x,p,u) \in T^*\mathbb{T} \times \mathbb{R} \mid H(x,p,u) = 0\}.$$

To verify Item (i), it suffices to consider the linearization of (B.1) in a neighborhood of $(x_2, 0) \in T^*\mathbb{T}$. It is formulated as

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} V'(x_2) & 1 \\ 0 & -(V'(x_2)+\lambda) \end{bmatrix} \begin{bmatrix} x - x_2 \\ p \end{bmatrix}. \tag{B.2}$$

By the assumptions on $V(x)$, Item (i) holds.

B.2. **On Item (ii).** This item was proved by [23, Proposition 1]. We omit it.

B.3. **On Item (iii).** To fix the notations, we use $\mathcal{D}_{v_+}$ to denote the set of differentiable points of v_+. For each $v_+ \in \mathcal{S}_+$, we know that it is semiconvex with linear modulus (see [3, Theorem 5.3.6]). Moreover, $\mathcal{D}_{v_+}$ has full Lebesgue measure on $\mathbb{T}$. Denote $\check{\mathcal{M}} := \Pi^*\tilde{\mathcal{M}}$, where $\Pi^* : T^*\mathbb{T} \times \mathbb{R} \to T^*\mathbb{T}$. Namely, $\check{\mathcal{M}}$ denotes the projection of the Mather set $\tilde{\mathcal{M}}$ to $T^*\mathbb{T}$. Let $\check{\Phi}_t := \Pi^*\Phi_t$.

The following lemma is from [7, Proposition 4.5].

Lemma B.1. *Let us consider*

$$\lambda u + \check{H}(x, d_x u) = c(\check{H}) \qquad \text{in } \mathbb{T}, \tag{B.3}$$

where $\check{H} : T^*\mathbb{T} \to \mathbb{R}$ is a C^3–Hamiltonian, satisfying Tonelli assumptions and $\mathbb{T}$ is a flat circle. Let $(x_0, 0) \in \check{\mathcal{M}}$ be a saddle point for the discounted flow generated by

$$\begin{cases} \dot{x} = \frac{\partial \check{H}}{\partial p}(x,p), \\ \dot{p} = -\frac{\partial \check{H}}{\partial x}(x,p) - \lambda p. \end{cases} \tag{DH}$$

Given $v_+ \in \mathcal{S}_+$ with $v_+(x_0) = u_-(x_0)$, let $\bar{x}_1, \bar{x}_2$ be two differentiable points of v_+ with $x_0 \in (\bar{x}_1, \bar{x}_2)$. Denote $\bar{p}_i := d_{\bar{x}_i} v_+$ with $i = 1, 2$. If

$$(x_0, 0) \in \omega(\bar{x}_1, \bar{p}_1) \cap \omega(\bar{x}_2, \bar{p}_2),$$

where $\omega(\bar{x}_i, \bar{p}_i)$ denotes the ω-limit set of $(\bar{x}_i, \bar{p}_i)$. Then there exists $\delta := \delta(v_+) > 0$ such that $d_x v_+$ exists for all $x \in [x_0 - \delta, x_0 + \delta]$, and the set

$$\{(x, d_x v_+) \mid x \in [x_0 - \delta, x_0 + \delta]\}$$

coincides with the local stable submanifold of $(x_0, 0)$.

Next, we prove Item (iii). Note that $u_- \equiv 0$ is the classical solution. Then $u_+ \equiv 0$ is the maximal forward weak KAM solution. By [23, Proposition 10], if the forward weak KAM solution is unique, then $\tilde{\mathcal{S}}_s = \tilde{\mathcal{A}}$. It follows from Proposition 2.11(ii) that there exists $v_+ \in \mathcal{S}_+$ different from u_+. Thus, $v_+ \leq 0$ and there exists $x_0 \in \mathbb{T}$ such that $v_+(x_0) < 0$. Consider

$$\mathcal{I} := \{x \in \mathbb{T} \mid v_+(x) = 0\}.$$

Then $\mathcal{I}$ is a compact invariant set by $\pi^*\Phi_t$, where $\pi^* : T^*\mathbb{T} \times \mathbb{R} \to \mathbb{T}$ denotes the standard projection. Denote a fundamental domain of $\mathbb{T}$ by $[x_1, x_1 + 1)$. Consequently, there are several possibilities for $\mathcal{I}$ restricting on $[x_1, x_1 + 1)$:

$$\{x_1\},\ \{x_2\},\ \{x_1, x_2\},\ [x_1, x_2],\ [x_2, x_1 + 1) \cup \{x_1\},\ [x_1, x_1 + 1). \tag{B.4}$$

The remaining proof is divided into two steps.

Step 1. For each $v_+ \in \mathcal{S}_+$, $v_+(x_2) < 0$.
We assert that if $v_+(x_2) = 0$, then $v_+ \equiv u_+ = 0$. In fact, by (B.4), it suffices to show that there exists $\varepsilon > 0$, such that $v_+ = 0$ on $[x_2 - \varepsilon, x_2 + \varepsilon]$. By contradiction, we assume there exists $\bar{x} \in \mathcal{D}_{v_+} \cap [x_2 - \frac{1}{2}\varepsilon, x_2)$, such that $v_+(\bar{x}) < 0$. Let $\bar{p} := d_{\bar{x}} v_+$.

Let $(x(t),p(t)) := \check{\Phi}_t(\bar{x},\bar{p})$ for all $t \geq 0$. Then one can find $t_0 \geq 0$ such that $x(t_0) \in [x_2-\varepsilon, x_2)$ and $d_x v$ exists at $x = x(t_0)$ with $d_{x(t_0)}v_+ > 0$. Note that

$$\dot{x}(t_0) = p(t_0) + V(x(t_0)) = d_{x(t_0)}v_+ + V(x(t_0)).$$

By the definition of $V(x)$, $V(x(t_0)) > 0$. It follows that $\dot{x}(t_0) > 0$. Thus, $(x_2, 0) \in \omega(\bar{x},\bar{p})$. By (B.2), if $\lambda < |V'(x_2)|$, $(x_2, 0)$ is a saddle point in $\check{\mathcal{M}}$. By Lemma B.1, $v_+ = 0$ on $[x_2-\varepsilon, 0]$. Similarly, we have $v_+ = 0$ on $[x_2-\varepsilon, x_2+\varepsilon]$.

By the assertion above, we have $v_+(x_2) < 0$ and $\mathcal{I} = \{x_1\}$.

Step 2. The forward weak KAM solution v_+ different from $u_+ \equiv 0$ is unique.
Let $\bar{x} \in \mathcal{D}_{v_+} \cap (x_1, x_2)$. By Proposition 2.11(ii),

$$\tilde{\mathcal{M}} = \{(x_1,0,0),\ (x_2,0,0)\}, \tag{B.5}$$

which together with $v_+(x_2) < 0$ implies $(x_1, 0) \in \omega(\bar{x},\bar{p})$. In fact, if $(x_1, 0) \notin \omega(\bar{x},\bar{p})$, then $(x_2, 0) \in \omega(\bar{x},\bar{p})$. Moreover, $v_+(x(t)) \to v_+(x_2) < 0$ as $t \to +\infty$, which contradicts (B.5). By (B.2), $(x_1, 0) \in \check{\mathcal{M}}$ is a saddle point. By Lemma B.1, for any $v_+ \in \mathcal{S}_+$, there exists $\delta := \delta(v_+) > 0$ such that $d_x v_+$ exists for all $x \in [x_1-\delta, x_1+\delta]$, and the set

$$\{(x, d_x v_+) \mid x \in [x_1-\delta, x_1+\delta]\}$$

coincides with the local stable submanifold of $(x_1, 0)$.

By contradiction, we assume there exists another $\bar{v}_+$ that is different from both u_+ and v_+. From the discussion above, we have $\bar{v}_+(x_1) = v_+(x_1) = 0$ and one can find $\delta' > 0$ such that $d_x\bar{v}_+ = d_x v_+$ on $[x_1-\delta', x_1+\delta']$. That yields $\bar{v}_+ = v_+$ on $[x_1-\delta', x_1+\delta']$. By [7, Proposition 4], $\bar{v}_+ = v_+$ on $\mathbb{T}$. Therefore, the forward weak KAM solution v_+ different from u_+ is unique.

B.4. **On Item (iv).** To verify Item (iv), we only need to construct an example to show that for λ large enough, $\mathcal{S}_+$ contains more than two elements. We choose $\lambda = 3$ and $V(x) = \sin x$. Then $x_1 = 0$, $x_2 = \pi$. Let

$$\varphi(x) := \cos x - 1, \quad \psi(x) := -\cos x - 1.$$

A direct calculation shows that

$$3\varphi + \frac{1}{2}|D\varphi|^2 + D\varphi \cdot \sin x \leq 0, \quad 3\psi + \frac{1}{2}|D\psi|^2 + D\psi \cdot \sin x \leq 0,$$

which means both φ and ψ are viscosity subsolutions of

$$3u(x) + \frac{1}{2}|Du|^2 + Du \cdot \sin(x) = 0, \qquad x \in \mathbb{T}. \tag{B.6}$$

Then

$$T_t^+\varphi \leq \varphi, \quad T_t^+\psi \leq \psi. \tag{B.7}$$

Note that $\varphi, \psi \leq 0$ and $\varphi(0) = \psi(\pi) = 0 = u_-(0)$. By [22, Proposition 13], both $T_t^+\varphi$ and $T_t^+\psi$ converge as $t \to +\infty$. Let

$$u_1(x) := \lim_{t\to+\infty} T_t^+\varphi, \quad u_2(x) := \lim_{t\to+\infty} T_t^+\psi.$$

Then $u_1, u_2 \in \mathcal{S}_+$. By (B.7) and the constructions of φ and ψ, we know that u_1, u_2 and $u_+ \equiv 0$ are different from each other.

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