A weakly coupled mean field games model of first order for k groups of major players

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Abstract

In this paper, we introduce a weakly coupled mean field games model of first order for k different kinds of major players. The existence of solutions of this kind of weakly coupled mean field games model is proved.

Keywords. weakly coupled systems, mean field games, existence

1 Introduction and main results

In the present paper, we consider a weakly coupled mean field games model of first order for k kinds of major players. The way they interact is not only the mean field terms, but also the value of the cost functional. Assume M is a connected, closed (compact without boundary) and smooth Riemannian manifold. For $x \in M$ and $i \in \{1, \dots, k\}$, let D denote the spacial gradient with respect to x, we consider the following model

$$\begin{cases} H_i(x, Du_i(x)) + \sum_{j=1}^{k} B_{ij}(x) u_j(x) = F_i(x, m_1, \dots, m_k), & (1.1) \\ \operatorname{div}\left(m_i \dfrac{\partial H_i}{\partial p}(x, Du_i(x)) \right) = 0, & (1.2) \\ \int_M m_i dx = 1. & (1.3) \end{cases}$$

In the classical mean field games model, there is a large community of identical players. The cost functional of each player is determined by the same Hamiltonian H. As pointed in [1], in most real problems of economics, there is not just one major player. So it is natural to consider a system with several major players. The model (1.1)-(1.3) describes a system with k groups of major players. The Hamiltonians of each group of players are different from each other. The cost functional of the ith major player is determined by the Hamiltonian H_i, the cost functional u_j of the jth major player, and the distribution m_j of the jth major player. The system is weakly coupled in the sense that the coupling terms depend only on the zero order terms, such as u_j and m_j, and do not depend on the gradient terms.

For each $i \in \{1, \dots, k\}$, the function $H_i : T^*M \to \mathbb{R}$ is of class C^2 and satisfies:

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(H1) (Positive definiteness). For every $(x,p) \in T^*M$, the second partial derivative $\frac{\partial^2 H_i}{\partial p^2}(x,p)$ is positive definite as a quadratic form.

(H2) (Superlinearity). For every $x \in M$, $H_i(x,p)$ is superlinear in p.

(H3) (Reversibility). $H_i(x,p) = H_i(x,-p)$ for every $(x,p) \in T^*M$.

The coupling matrix $(B_{ij}(x))$ is of class C^2 on M and satisfies

(B) The matrix $(B_{ij}(x))$ is irreducible (i.e., the system (1.1) can not be decoupled) and

$$B_{ij}(x) \le 0 \text{ if } i \neq j, \quad \sum_{j=1}^{k} B_{ij}(x) > 0 \text{ for all } i \in \{1,\dots,k\}.$$

In this paper, we denote by $C(M)$ (resp. $C(M,\mathbb{R}^k)$) the space of continuous functions from M to $\mathbb{R}$ (resp. $\mathbb{R}^k$). We also denote by $\|\cdot\|_\infty$ the supremum norm of both real and vector valued functions on its domain.

Let $\mathcal{P}(M)$ and $\mathcal{P}(T^*M)$ denote the set of Borel probability measures on M and T^*M respectively. Both of them are compact under the w^*-topology. Let X denotes either M or T^*M. A sequence $\{\mu_k\}_{k\in\mathbb{N}} \subset \mathcal{P}(X)$ is w^*-convergent to $\mu \in \mathcal{P}(X)$ if

$$\lim_{k\to+\infty} \int_X f(x)d\mu_k = \int_X f(x)d\mu, \quad \forall f \in C_b(X),$$

where $C_b(X)$ stands for the space of bounded and uniformly continuous functions on X. We shall work with the Monge-Wasserstein distance d_1, which is defined by

$$d_1(m_1,m_2) = \sup_h \int_M hd(m_1 - m_2), \quad \forall m_1,\ m_2 \in \mathcal{P}(X),$$

where the supremum is taken over all 1-Lipschitz continuous functions on X. We recall that d_1 metricizes the w^*-topology. In the following, the product space $\mathcal{P}(X)^k$ is endowed with the product topology. Then $\mathcal{P}(X)^k$ is compact by the Tychonoff theorem, and is endowed with a distance d^k induced from d_1. More precisely, for $\mathbf{m}^1 = (m_1^1,\dots,m_k^1)$ and $\mathbf{m}^2 = (m_1^2,\dots,m_k^2) \in \mathcal{P}(M)^k$, we define

$$d^k(\mathbf{m}_1,\mathbf{m}_2) = \max_{1\le i\le k} d_1(m_i^1,m_i^2).$$

The topology induced by d^k coincides with the product topology. For each $i \in \{1,\dots,k\}$, the function $F_i : M \times \mathcal{P}(M)^k \to \mathbb{R}$ satisfies:

(F1) for every $\mathbf{m} = (m_1,\dots,m_k) \in \mathcal{P}(M)^k$, the function $x \mapsto F_i(x,\mathbf{m})$ is of class C^2 and

$$\sup_{\mathbf{m}} \sum_{|\alpha|\le 1} \|D^\alpha F_i(\cdot,\mathbf{m})\|_\infty < +\infty,$$

where the index $\alpha = (\alpha_1,\dots,\alpha_n)$ and $D^\alpha = D^{\alpha_1}\dots D^{\alpha_n}$.

(F2) for every $x \in M$ the function $\mathbf{m} \mapsto F_i(x,\mathbf{m})$ is Lipschitz continuous and

$$\sup_{\substack{x\in M \\ \mathbf{m}_1,\mathbf{m}_2\in\mathcal{P}(M)^k \\ \mathbf{m}_1\neq\mathbf{m}_2}} \frac{|F_i(\cdot,\mathbf{m}_1) - F_i(\cdot,\mathbf{m}_2)|}{d^k(\mathbf{m}_1,\mathbf{m}_2)} < +\infty.$$

Definition 1.1. *Assume that $H_i : T^*M \times \mathbb{R}^k \to \mathbb{R}$ is continuous for each $i \in \{1, \dots, k\}$. A continuous function $\boldsymbol{u} : M \to \mathbb{R}^k$ is called a viscosity subsolution (resp. supersolution) of*

$$H_i(x, Du_i, \boldsymbol{u}) = 0, \quad x \in M, \ 1 \le i \le k, \tag{1.4}$$

if for each component u_i of $\boldsymbol{u}$ and test function ϕ of class C^1, when $u_i - \phi$ attains its local maximum (resp. minimum) at x, then

$$H_i(x, D\phi(x), \boldsymbol{u}(x)) \le 0, \quad (\text{resp. } H_i(x, D\phi(x), \boldsymbol{u}(x)) \ge 0).$$

A function is called a viscosity solution of (1.4) if it is both a viscosity subsolution and a viscosity supersolution.

Definition 1.2. *A solution of the system (1.1)-(1.3) is a couple $(\boldsymbol{u}, \boldsymbol{m}) \in C(M, \mathbb{R}^k) \times \mathcal{P}(M)^k$ such that (1.1) is satisfied in viscosity sense for weakly coupled systems of Hamilton-Jacobi equations and (1.2) is satisfied in distributions sense.*

Remark 1.1. *Assumptions (H1), (H2) are the classical Tonelli conditions. When the coupling matrix satisfies the condition (B), then for each $\boldsymbol{m} \in \mathcal{P}(M)^k$, the viscosity solution of the weakly coupled system (1.1) is unique by [9, Proposition 2.10]. The existence of the viscosity solution of the weakly coupled system (1.1) for fixed $\boldsymbol{m} \in \mathcal{P}(M)^k$ is guaranteed by the Perron's method for weakly coupled systems, see [24] for instance.*

Theorem 1. *Assume (H1)-(H3), (B) and (F1)(F2), then the system (1.1)-(1.3) admits at least one solution.*

Remark 1.2. *Comparing to [21], the dynamical meaning of the solution of the system (1.1)-(1.3) is weaker. Fixing $\boldsymbol{m} \in \mathcal{P}(M)^k$, there is a unique viscosity solution $(u_1^m, \dots, u_k^m)$ of (1.1). The viscosity solution is Lipschitz continuous by (H2). The function u_i^m is the unique viscosity solution of $H_i^m(x, Du, u) = 0$, where*

$$H_i^m(x, p, u) := H_i(x, p) + B_{ii}(x)u + \sum_{j=1, j\neq i}^{k} B_{ij}(x)u_j^m(x) - F_i(x, \boldsymbol{m}) \tag{1.5}$$

is only Lipschitz continuous in x. Therefore, the contact Hamiltonian flow of H_i^m is not well-defined. Following [11], the Mather measures can be defined as the closed Borel probability measures supported in the Mather set, see Definition 2.1 below. If $(\boldsymbol{u}, \boldsymbol{m})$ is a solution of (1.1)-(1.3), then for each $i \in \{1, \dots, k\}$, there is a Mather measure corresponding to $H_i^m(x, p, u)$ whose projection on M equals m_i.

The mean field games was introduced by Lasry and Lions in [28–30] and by Caines, Huang and Malhamé [22, 23] to analyze large population stochastic differential games. For the theory of first order mean field games, we refer to [4–7, 14, 17, 20]. For the theory of stationary mean field games, we refer to [2, 16, 18, 19, 32].

We denote by $\mathbb{T}^n$ the n-dimensional flat torus. Let $G : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ be of class C^2 and quadratic-like in the second variable. The following ergodic first order mean field game system with $x \in \mathbb{T}^n$ was discussed in [4]

$$\begin{cases} G(x, Du(x)) = c + F(x, m), \\ -\operatorname{div}\left(m \dfrac{\partial G}{\partial p}(x, Du(x))\right) = 0, \\ \displaystyle\int_{\mathbb{T}^n} m dx = 1. \end{cases}$$

The system is a Hamilton-Jacobi equation coupled with a continuity equation. The scalar unknown function u is defined on M, and the unknown m is a Borel probability measure define on M. The function F is a coupling between the two equations. The function u can be understood as the value of the cost functional or the objective functional of a typical small player. The optimal feedback of this small player is then given by $-\frac{\partial G}{\partial p}(x, Du(x))$. When all players play according to this rule, their density of distribution m is governed by the second continuity equation.

The following ergodic first order mean field game system of contact type was discussed in [21]

$$\begin{cases} H(x, Du(x), u(x)) = F(x, m), \\ \text{div}\left(m\dfrac{\partial H}{\partial p}(x, Du(x), u(x))\right) = 0, \\ \displaystyle\int_M m dx = 1. \end{cases}$$

Here $x \in M$ and M is a compact connected and smooth Riemannian manifold without boundary. The function $H : T^*M \times \mathbb{R} \to \mathbb{R}$ satisfies the standard dynamical assumptions, i.e., of class C^3, superlinear and strictly convex in p. The solution (u, m) of the system obtained in [21] has a clear dynamical meaning. More precisely, there is a Mather measure of the contact Hamilton flow generated by $H(x, p, u)$ whose projection on M equals m. The dynamical tools used in [21] come from [35].

The system (1.4) is weakly coupled in the sense that every ith equation depends only on Du_i, but not on Du_j for $j \neq i$. The standard assumption for (1.4) is so-called the monotonicity condition, which means H_i is increasing in u_i and nonincreasing in u_j for $j \neq i$. More precisely, for any $(x, p) \in T^*M$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^k$, if $u_l - v_l = \max_{1\leq i\leq k}(u_i - v_i) \geq 0$, then $H_l(x, p, \mathbf{u}) \geq H_l(x, p, \mathbf{v})$. When the coupling is linear, that is, when H_i has the form

$$H_i(x, p, \mathbf{u}) = h_i(x, p) + \sum_{j=1}^{k} B_{ij}(x)u_j,$$

the monotonicity condition holds if and only if

$$B_{ij}(x) \leq 0 \text{ if } i \neq j \quad \text{and} \quad \sum_{j=1}^{k} B_{ij}(x) \geq 0 \text{ for all } i \in \{1, \dots, k\}.$$

By [9, Proposition 1.2], if the coupling matrix $(B_{ij}(x))$ is irreducible, then $B_{ii}(x) > 0$ for every $i \in \{1, \dots, k\}$. About the stationary weakly coupled systems, there are several topics of concern. For the existence theorems and the comparison results of viscosity solutions, one can refer to [12, 24, 31]. For the weak KAM theory, one can refer to [9]. For the vanishing discount problem, one can refer to [10, 25, 26].

Remark 1.3. *When the coupling term $\sum_j B_{ij}u_j$ vanishes, one may consider the following coupled system for $i \in \{1 \dots, k\}$*

$$\begin{cases} H_i(x, Du_i(x)) = c_i + F_i(x, m_1, \dots, m_k), & (1.6) \\ \textit{div}\left(m_i\dfrac{\partial H_i}{\partial p}(x, Du_i(x))\right) = 0, & (1.7) \\ \displaystyle\int_M m_i dx = 1. & (1.8) \end{cases}$$

From the view of differential games, the major players influence each other only through the mean field terms in this case. The difficulty mentioned in Remark 1.2 does not appear. The proof of the existence of solutions $(\boldsymbol{c}, \boldsymbol{u}, \boldsymbol{m})$ is quite similar to the one in [4]. We provide an outline. Consider the set-vauled map $\Psi : \mathcal{P}(M)^k \rightrightarrows$

$\mathcal{P}(M)^k$. The set $\Psi(\boldsymbol{m}) := \{(\mu_1,\dots,\mu_k)\}$, where each μ_i is a projected Mather measures corresponding to $H_i(x,Du_i(x)) - F_i(x,\boldsymbol{m})$. Since the products of compact convex sets are still compact and convex, it remains to check Ψ has a closed graph. This property is guaranteed by (H2) and (F2). Thus, Ψ admits a fixed point $\bar{\boldsymbol{m}}$ by the Kakutani's fixed point theorem. For $\bar{\boldsymbol{m}} = (\bar{m}_1,\dots,\bar{m}_k) \in \mathcal{P}(M)^k$, there exists a solution (u_i,c_i) of (1.6) by the classical theory of Hamilton-Jacobi equations, and the probability measure $\bar{m}_i$ naturally satisfies (1.7). When all H_i and F_i coincide, the system (1.6)-(1.8) can be reduced to the case considered in [4].

2 Proof of the main theorem

By [3, Corollary A.2.7], when $H_i : T^*M \to \mathbb{R}$ satisfies (H1)(H2), the corresponding Lagrangian $L_i : TM \to \mathbb{R}$ is also Tonelli, i.e., of class C^2, superlinear and strictly convex in the fibre. At the present, we denote by $H(x,p,u)$ the Hamiltonian defined in (1.5) for simplicity of notation, then it satisfies the basic assumptions in [27], and is strictly increasing in u. Let $\lambda := \max_{1\le i\le k} \|B_{ii}(x)\|_\infty$ be the Lipschitz constant for $H(x,p,u)$ in u. The corresponding Lagrangian associated to H is

$$L(x,\dot{x},u) = L_i(x,\dot{x}) - B_{ii}(x)u - \sum_{j=1,j\neq i}^{k} B_{ij}(x)u_j^m(x) + F_i(x,\mathbf{m}), \tag{2.1}$$

The following proposition holds when $H_i(x,p)$ in (1.5) satisfies (H1)(H2).

Proposition 2.1. *Let (u_-,u_+) be the conjugate pair defined in Proposition A.6, and $\mathcal{I}_{(u_-,u_+)}$ be the corresponding projected Aubry set. For $x \in \mathcal{I}_{(u_-,u_+)}$, there exists a C^1 curve $\gamma : (-\infty,\infty) \to M$ with $\gamma(0) = x$ such that $u_-(\gamma(t)) = u_+(\gamma(t))$, and*

$$u_\pm(\gamma(t')) - u_\pm(\gamma(t)) = \int_t^{t'} L(\gamma(s),\dot{\gamma}(s),u_\pm(\gamma(s)))ds, \quad \forall t \le t' \in \mathbb{R}. \tag{2.2}$$

In addition, $u_\pm$ are differentiable at x with the same derivative

$$Du_\pm(x) = \frac{\partial L}{\partial \dot{x}}(x,\dot{\gamma}(0),u_\pm(x)). \tag{2.3}$$

We then define

$$\tilde{\mathcal{I}}_{(u_-,u_+)} := \{(x,p,u) :\ x \in \mathcal{I}_{(u_-,u_+)},\ p = Du_\pm(x),\ u = u_\pm(x)\}.$$

Both u_- and u_+ are of class C^1 on $\mathcal{I}_{(u_-,u_+)}$, or equivalently, the lift from $\mathcal{I}_{(u_-,u_+)}$ to $\tilde{\mathcal{I}}_{(u_-,u_+)}$ is continuous.

We divide the proof of Proposition 2.1 into Lemmas 2.1-2.4. In the following, we denote by $d(x,y)$ the distance between x and y induced by the Riemannian metric g on M. We also denote by $|\cdot|_x$ the norms induced by the Riemannian metric g on both tangent and cotangent spaces of M.

Lemma 2.1. *If $u \prec L$, then u is a Lipschitz continuous function defined on M.*

Proof. For each $x,y \in M$, let $\alpha : [0,d(x,y)] \to M$ be a geodesic of length $d(x,y)$, with constant speed $|\dot{\alpha}|_\alpha = 1$ and connecting x and y. Let C_L denote the bound of $L(x,\dot{x},0)$ for $|\dot{x}|_x \le 1$. Then

$$L(\alpha(s),\dot{\alpha}(s),u(\alpha(s))) \le C_L + \lambda\|u\|_\infty, \quad \forall s \in [0,d(x,y)].$$

Then by $u \prec L$ we have

$$u(y) - u(x) \leq \int_0^{d(x,y)} L(\alpha(s), u(\alpha(s)), \dot\alpha(s))ds \leq (C_L + \lambda\|u\|_\infty)d(x,y).$$

Exchanging the role of x and y, we get the Lipschitz continuity of u. □

Lemma 2.2. *Given $a > 0$. If $u \prec L$, let $\gamma : [-a, a] \to M$ be a $(u, L, 0)$-calibrated curve, then γ is of class C^1 and u is differentiable at $\gamma(0)$.*

Proof. By Lemma 2.1, for each $i \in \{1, \ldots, k\}$, the backward weak KAM solution u_i^m is Lipschitz continuous. Thus, $H(x, p, u) = H_i^m(x, p, u)$ is locally Lipschitz continuous in x. Since the following argument is local, we identify TM with an open subset of $\mathbb{R}^n \times \mathbb{R}^n$. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. For $\|v_1\|$ and $\|v_2\|$ less than R, there exists a constant $K > 0$ such that

$$\begin{aligned}&|L(x_1, v_1, u(x_1)) - L(x_2, v_2, u(x_2))|\\ &\leq |L(x_1, v_1, u(x_1)) - L(x_1, v_1, u(x_2))| + |L(x_1, v_1, u(x_2)) - L(x_2, v_2, u(x_2))|\\ &\leq \lambda\|Du(x)\|_\infty d(x_1, x_2) + K(d(x_1, x_2) + \|v_1 - v_2\|).\end{aligned}$$

Therefore $(x, \dot x) \mapsto L(x, \dot x, u(x))$ is locally Lipschitz continuous. Note that $L(x, \dot x, u)$ is strictly convex in $\dot x$, by [8, Theorem 2.1 (ii)], the minimizer γ is a C^1 curve.

The following argument is similar to [35, Lemma 4.3]. Since we are arguing locally near the point $x := \gamma(0)$, it suffices to prove the lemma for the case when M is an open subset U of $\mathbb{R}^n$. We are going to show for each $y \in U$, there holds

$$\limsup_{\eta\to 0^+} \frac{u(x + \eta y) - u(x)}{\eta} \leq \frac{\partial L}{\partial \dot x}(x, \dot\gamma(0), u(x)) \cdot y \leq \liminf_{\eta\to 0^+} \frac{u(x + \eta y) - u(x)}{\eta}. \tag{2.4}$$

For $\eta > 0$ and $0 < \varepsilon \leq a$, define $\gamma_\eta : [-\varepsilon, 0] \to U$ by $\gamma_\eta(s) = \gamma(s) + \frac{s+\varepsilon}{\varepsilon}\eta y$, then $\gamma_\eta(0) = x + \eta y$ and $\gamma_\eta(-\varepsilon) = \gamma(-\varepsilon)$. Since γ is a $(u, L, 0)$-calibrated curve

$$u(x + \eta y) - u(\gamma(-\varepsilon)) \leq \int_{-\varepsilon}^0 L(\gamma_\eta(s), \dot\gamma_\eta(s), u(\gamma_\eta(s)))ds,$$

$$u(x) - u(\gamma(-\varepsilon)) = \int_{-\varepsilon}^0 L(\gamma(s), \dot\gamma(s), u(\gamma(s)))ds.$$

It follows that

$$\frac{u(x + \eta y) - u(x)}{\eta} \leq \frac{1}{\eta}\int_{-\varepsilon}^0 (L(\gamma_\eta(s), \dot\gamma_\eta(s), u(\gamma_\eta(s))) - L(\gamma(s), \dot\gamma(s), u(\gamma(s))))ds.$$

By the locally Lipschitz continuity of the map $(x, \dot x) \mapsto L(x, \dot x, u(x))$, there exists $K'(\|\dot\gamma(s)\|)$ such that

$$\begin{aligned}&\limsup_{\eta\to 0^+}\frac{u(x + \eta y) - u(x)}{\eta}\\ &\quad\leq \limsup_{\eta\to 0^+} \frac{1}{\eta}\int_{-\varepsilon}^0 (L(\gamma_\eta(s), \dot\gamma_\eta(s), u(\gamma_\eta(s))) - L(\gamma_\eta(s), \dot\gamma(s), u(\gamma_\eta(s))))\\ &\qquad + (L(\gamma_\eta(s), \dot\gamma(s), u(\gamma_\eta(s))) - L(\gamma(s), \dot\gamma(s), u(\gamma(s))))ds\\ &\quad\leq \int_{-\varepsilon}^0 (\frac{1}{\varepsilon}\frac{\partial L}{\partial \dot x}(\gamma(s), \dot\gamma(s), u(\gamma(s))) \cdot y + K'(\|\dot\gamma(s)\|)\frac{s+\varepsilon}{\varepsilon}\|y\|)ds.\end{aligned}$$

Let $\varepsilon \to 0^+$, we get the first equality in (2.4). Similarly, define $\gamma_\eta : [0,\varepsilon] \to U$ by $\gamma_\eta(s) = \gamma(s) + \frac{\varepsilon - s}{\varepsilon}\eta y$, we get the second equality in (2.4). □

Lemma 2.3. *Given a conjugate pair (u_-, u_+), for $x \in \mathcal{I}_{(u_-,u_+)}$, there exists a C^1 curve $\gamma : (-\infty,\infty) \to M$ with $\gamma(0) = x$ such that $u_-(\gamma(t)) = u_+(\gamma(t))$, and*

$$u_\pm(\gamma(t')) - u_\pm(\gamma(t)) = \int_t^{t'} L(\gamma(s), \dot\gamma(s), u_\pm(\gamma(s)))ds, \quad \forall t \le t' \in \mathbb{R}. \tag{2.5}$$

In addition, $u_\pm$ are differentiable at x with the same derivative.

Proof. For $x \in \mathcal{I}_{(u_-,u_+)}$, there is a $(u_-, L, 0)$-calibrated curve $\gamma_- : (-\infty, 0] \to M$ with $\gamma_-(0) = x$ and a $(u_+, L, 0)$-calibrated curve $\gamma_+ : [0, +\infty) \to M$ with $\gamma_+(0) = x$, connecting these two curves, we get a curve $\gamma : (-\infty,\infty) \to M$ with $\gamma(0) = x$.

According to the proof of [35, Lemma 4.7], we have $u_+(\gamma_+(s)) = u_-(\gamma_+(s))$ for $s \ge 0$ and $u_+(\gamma_-(s)) = u_-(\gamma_-(s))$ for $s \le 0$. Therefore γ is a $(u_\pm, L, 0)$-calibrated curve defined on the whole $\mathbb{R}$, i.e. the equality (2.5) holds. By Lemma 2.2, γ is a C^1 curve and $Du_\pm(x) = \frac{\partial L}{\partial \dot x}(x, u_\pm(x), \dot\gamma(0))$. □

Lemma 2.4. *The conjugate pair u_- and u_+ are both of class C^1 on $\mathcal{I}_{(u_-,u_+)}$.*

Proof. By Lemma 2.1, u_- is Lipschitz continuous. By [3, Theorem 5.3.7], if $H(x, p, u)$ is locally Lipschitz continuous and stictly convex in p, then u_- is locally semiconcave. Similarly, since $-u_+$ is a viscosity solution of (A.3), it is also locally semiconcave. Equivalently u_+ is locally semiconvex. Then by [3, Theorem 3.3.7], the conjugate pair u_- and u_+ are both C^1 on $\mathcal{I}_{(u_-,u_+)}$. □

Definition 2.1. *Given a conjugate pair (u_-, u_+). The projected Mather set $\mathcal{M}_H$ is defined as the limit set of the calibrated curves passing through the points in $\mathcal{I}_{(u_-,u_+)}$. By (2.3), we have*

$$\dot\gamma(0) = \frac{\partial H}{\partial p}(x, Du_\pm(x), u_\pm(x)). \tag{2.6}$$

Since the viscosity solution u_- is unique, and H is strictly convex in p, $\dot\gamma(0)$ is uniquely determined by the position x. We then define the Mather set

$$\tilde{\mathcal{M}}_H := \{(x, v) : \; x \in \mathcal{M}_H, \; v = \frac{\partial H}{\partial p}(x, Du_\pm(x), u_\pm(x))\} \subset TM.$$

Since $Du_\pm(x)$ is just continuous, the solution of the following ordinary differential equation

$$\dot x = \frac{\partial H}{\partial p}(x, Du_\pm(x), u_\pm(x))$$

may not be unique. Thus, we can not construct a flow on $\mathcal{I}_{(u_-,u_+)}$ by γ. We define the Mather measures via the closed measures. A closed measure μ on TM is defined by

$$\int_{TM} |v|_x d\mu(x, v) < +\infty \quad \text{and} \quad \int_{TM} d_x\varphi(v) d\mu(x, v) = 0 \quad \forall \varphi \in C^1(M).$$

Here d_x denotes the exterior differential with respect to x. The Mather measure can be defined as the closed Borel probability measures supported in $\tilde{\mathcal{M}}_H$. Denote by $\mathfrak{M}_H$ the set of Mather measures. One can construct Mather measures by closed calibrated curves in $\mathcal{I}_{(u_-,u_+)}$.

From now on, we denote by $H_i^m : T^*M \times \mathbb{R} \to \mathbb{R}$ the Hamiltonian defined as in (1.5), with $H_i(x,p)$ satisfies (H1)-(H3). The unique viscosity solution of $H_i^m(x,Du,u)=0$ is denoted by u_i^m. The corresponding projected Abury set, Mather set, projeted Mather set and the set of Mather measures are denoted by $\mathcal{A}_i^m$, $\tilde{\mathcal{M}}_i^m$, $\mathcal{M}_i^m$ and $\mathfrak{M}_i^m$ respectively.

Lemma 2.5. *For any $\boldsymbol{m} \in \mathcal{P}(M)^k$, define*

$$\mathcal{K}_i^m := \{(x,0) \in TM : \ H_i^m(x,0,u_i^m(x)) = 0\}.$$

Then $\mathcal{K}_i^m$ is a nonempty compact subset of the Mather set $\tilde{\mathcal{M}}_i^m$ and all points in it is fixed. Thus, the Mather measures exist. Moreover, we have $\mathcal{K}_i^m = \tilde{\mathcal{M}}_i^m$.

Proof. The proof is quite similar to the proof of [21, Proposition 7 and 8]. Let $(x,0) \in \mathcal{K}_i^m$. By (H3), the Lagrangian corresponding to H_i^m satisfies

$$\begin{aligned} L_i^m(x,0,u_i^m(x)) &= \sup_{p \in T_x^*M} (-H_i^m(x,p,u_i^m(x))) \\ &= -\inf_{p \in T_x^*M} H_i^m(x,p,u_i^m(x)) = H_i^m(x,0,u_i^m(x)) = 0. \end{aligned}$$

Let $\gamma_x(s) \equiv x$ for $s \in (-\infty, 0]$. Then for all $t<0$ we have

$$u_i^m(x) - u_i^m(\gamma_x(t)) = \int_t^0 L_i^m(\gamma_x(s), \dot\gamma_x(s), u_i^m(\gamma_x(s)))ds = \int_t^0 L_i^m(x,0,u_i^m(x))ds = 0.$$

Thus, γ^x is a $(u_i^m, L_i^m, 0)$-calibrated curve. By Proposition A.4, we have $T_t^+ u_i^m(x) = u_i^m(x)$. By Proposition A.5, the limit $\lim_{t\to+\infty} T_t^+ u_i^m(x)$ exists, and equals a forward weak KAM solution v_i^m of $H_i^m(x,Du,u)=0$. The pair (u_i^m, v_i^m) is a conjugate pair. Let $t \to +\infty$, we have $v_i^m(x) = u_i^m(x)$, i.e., $x \in \mathcal{A}_i^m$. By definition of the Mather set, we have $(x,0) \in \tilde{\mathcal{M}}_i^m$.

Since H_i^m is locally Lipschitz continuous, and strictly convex in p, the viscosity solution u_i^m is semiconcave on M by [3, Theorem 5.7]. Let x_0 be a minimal point of u_i^m, then u_i^m is differentiable at x_0 and $Du_i^m(x_0)=0$. Since u_i^m is a viscosity solution, we have $H_i^m(x_0,0,u_i^m(x_0))=0$. Therefore, $\mathcal{K}_i^m$ is nonempty.

It remains to proof $\mathcal{K}_i^m \supset \tilde{\mathcal{M}}_i^m$. Let $\gamma(s)$ be a arbitrary trajectory contained in $\mathcal{M}_i^m$, we want to show it is a fixed point. By the C^1-regularity of $u_m^i|_{\mathcal{A}_i^m}$ and $\gamma(s)$, combining with (2.6), we have

$$\frac{d}{ds} u_i^m(\gamma(s)) = \langle Du_i^m(\gamma(s)), \dot\gamma(s)\rangle = \langle Du_i^m(\gamma(s)), \frac{\partial H_i}{\partial p}(\gamma(s), Du_i^m(\gamma(s)))\rangle.$$

Since $\mathcal{M}_i^m$ is a limit set, the recurrence property of points in the Mather set still holds. The assumption (H3) implies $\langle p, \partial H_i/\partial p\rangle \geq 0$, and the equality holds if and only if $p=0$. If $Du_i^m(\gamma(s)) \neq 0$, then $du_i^m(\gamma(s))/ds > 0$, which contradicts the recurrence property of $\mathcal{M}_i^m$. Thus, we have $Du_i^m(\gamma(s))=0$, which implies that $\dot\gamma(s)=0$. Finally, we conclude that all points in $\tilde{\mathcal{M}}_i^m$ have the form $(x,0)$, and satisfy $H_i^m(x,0,u_i^m(x))=0$. □

Lemma 2.6. *For any $\boldsymbol{m} \in \mathcal{P}(M)^k$, let $\boldsymbol{u}^m$ denote the unique viscosity solution of (1.1). Then $\boldsymbol{u}^m$ is uniformly bounded and equi-Lipschitz with respect to $\boldsymbol{m}$. Let $\boldsymbol{m}_j \in \mathcal{P}(M)^k$ converges to $\boldsymbol{m}_0$ in the sense of d^k, then $\boldsymbol{u}^{m_j}$ converges to $\boldsymbol{u}^{m_0}$ uniformly.*

Proof. By (B)(F1) and the boundedness of $H_i(x,0)$, there is a constant C large enough such that $(C,\dots,C)$ and $(-C,\dots,-C)$ are a supersolution and a subsolution of (1.1) for any $\mathbf{m}\in\mathcal{P}(M)^k$. By the comparison principle [9, Proposition 2.10], the constant C is the uniform bound of $\mathbf{u}^m$. The Lagrangian $L(x,\dot{x},u)$ corresponding to $H_i^m(x,p,u)$, which is given by (2.1), is also uniformly bounded for $|\dot{x}|_x\leq 1$ with respect to $\mathbf{m}$. Using a similar argument as in Lemma 2.1, one can prove that $\mathbf{u}^m$ is equi-Lipschitz with respect to $\mathbf{m}$. By the Arzelá-Ascoli theorem, there is a subsequence $\mathbf{u}^{m_{j_k}}$ uniformly converges, with the limit point $\mathbf{u}^*$. Let H_i^m be defined as in (1.5). By (F2), $H_i^{m_{j_k}}$ converges uniformly to

$$H_i^*(x,p,u):=H_i(x,p)+B_{ii}(x)u+\sum_{j=1,j\neq i}^{k}B_{ij}(x)u_j^*(x)-F(x,\mathbf{m}_0)$$

on compact subsets of $T^*M\times\mathbb{R}$. By the stability of viscosity solutions, $\mathbf{u}^*$ solves (1.1) with $(m_1,\dots,m_k)=\mathbf{m}_0$. By the uniqueness of the viscosity solution of (1.1) under the assumption (B), all limit points of $\{\mathbf{u}^m\}$ equals $\mathbf{u}^{m_0}$. □

Proof of Theorem 1. By the Prokhorov's theorem and the Tychonoff theorem, $(\mathcal{P}(M)^k,d^k)$ is compact and convex. Let $\pi: TM\to M$ be the canonical projection, which induces the push forward $\pi_\#$. Define the set-vauled map

$$\Psi:\mathcal{P}(M)^k\rightrightarrows\mathcal{P}(M)^k,$$

where we define the set

$$\Psi(\mathbf{m}):=\{(\pi_\#\eta_1^m,\dots,\pi_\#\eta_k^m):\ \eta_i^m\in\mathfrak{M}_i^m\}.$$

One can easily check that Ψ has nonempty convex values by Lemma 2.5. In order to use the Kakutani's fixed point theorem, it remains to show that Ψ has a closed graph. Let $d^k(\mathbf{m}_j,\mathbf{m})\to 0$ and $d^k(\mu^j,\mu)\to 0$ as $j\to+\infty$, where $\mu^j\in\Psi(\mathbf{m}_j)$. We want to show $\mu\in\Psi(\mathbf{m})$.

Since $\mu^j\in\Psi(\mathbf{m}_j)$, there exist a sequence η^j with $\eta_i^j\in\mathfrak{M}_i^{m_j}$ such that $\mu_i^j=\pi_\#\eta_i^j$. By Lemmas 2.5, we have

$$\text{supp}(\eta_i^j)\subset M\times\{0\}=:K_0,\quad \forall i\in\{1,\dots,k\},\ j\in\mathbb{N},$$

where supp stands for the support of Borel probability measures. Thus, the sequence η_i^j is tight. Up to a subsequence if necessary, we may suppose that $d^k(\eta^j,\eta)\to 0$ for some $\eta\in\mathcal{P}(M)^k$ and $\mu_i=\pi_\#\eta_i$ for each $i\in\{1,\dots,k\}$.

Now we show that $\eta_i\in\mathfrak{M}_i^m$, which implies that $\mu\in\Psi(\mathbf{m})$. We first show that η_i is closed. The integral $\int_{TM}|v|_x d\eta_i$ is finite by the compactness of K_0. By definition of K_0 we have

$$\int_{TM}d_x\varphi(v)d\eta_i=\int_{K_0}d_x\varphi(v)d\eta_i=0,\quad \forall\varphi\in C^1(M),$$

which shows that η_i is closed.

Next, we show that $\text{supp}(\eta_i)\subset\mathcal{K}_i^m$. Since η_i^j converges to η_i in the w^*-topology, for any $(x_0,v_0)\in\text{supp}(\eta_i)$, there is a sequence $(x_j,v_j)\in\text{supp}(\eta_i^j)$ converging to it. By Lemma 2.5 and $\eta_i^j\in\mathfrak{M}_i^{m_j}$, we have $v_j=0$ and

$$\begin{aligned}H_i^{m_j}(x_j,0,u_i^{m_j}(x_j))=&H_i(x_j,0)+B_{ii}(x_j)u_i^{m_j}(x_j)\\&+\sum_{j=1,j\neq i}^{k}B_{ij}(x_j)u_j^{m_j}(x_j)-F(x_j,\mathbf{m}_j)=0.\end{aligned}$$

Let $j \to +\infty$, by Lemma 2.6 and (F1)(F2), we get that $v_0 = 0$ and $H_i^m(x_0, 0, u_i^m(x_0)) = 0$. Therefore, Ψ has a closed graph. We conclude that Ψ admits a fixed point $\bar{\mathbf{m}}$.

Let $\bar{\mathbf{u}}(x) = (\bar{u}_1(x), \ldots, \bar{u}_k(x))$ be the unique viscosity solution of the weakly coupled system (1.1) with $(m_1, \ldots, m_k) = \bar{\mathbf{m}}$. Then there is a Mather measure $\bar{\eta}_i$ such that $\pi_\# \bar{\eta}_i = \bar{m}_i$. Since $\bar{\eta}_i$ is closed, for all $\varphi \in C^1(M)$ we have

$$
\begin{aligned}
0 &= \int_{TM} d_x\varphi(v) d\bar{\eta}_i = \int_{\text{supp}(\bar{\eta}_i)} d_x\varphi(v) d\bar{\eta}_i \\
&= \int_{\text{supp}(\bar{m}_i)} \langle D\varphi(x), \frac{\partial H_i}{\partial p}(x, D\bar{u}_i(x)) \rangle d\bar{m}_i = \int_M \langle D\varphi(x), \frac{\partial H_i}{\partial p}(x, D\bar{u}_i(x)) \rangle d\bar{m}_i.
\end{aligned}
$$

Hence, $\bar{\mathbf{m}}$ satisfies (1.2) in the sense of distribution. The proof is now complete. □

A Facts on the contact Hamilton-Jacobi equation

In this section, we collect some facts given by [27] in view of the contact Hamiltonian $H : T^*M \times \mathbb{R} \to \mathbb{R}$ defined by (1.5). As mentioned in Remark 1.2, we have to deal with Hamiltonians which are just Lipschitz continuous in x. Similar results can be founded in [34, 35] for C^3 Hamiltonians. Let us consider the evolutionary equation:

$$
\begin{cases}
\partial_t u(x,t) + H(x, Du(x,t), u(x,t)) = 0, & (x,t) \in M \times (0, +\infty). \\
u(x, 0) = \varphi(x), & x \in M.
\end{cases}
\tag{A.1}
$$

and the stationary equation:

$$
H(x, Du(x), u(x)) = 0. \tag{A.2}
$$

Since $H(x, p, u)$ defined by (1.5) is strictly increasing in u, the viscosity solution of (A.2) is unique by the comparison principle.

Proposition A.1. *[27, Theorem 3.1] Let the initial data $\varphi(x) \in LSC(M, \mathbb{R} \cup \{+\infty\})$, where $LSC(M, \mathbb{R} \cup \{\infty\})$ denotes the set of lower semi-continuous functions, with values taken in $\mathbb{R} \cup \{+\infty\}$. Then the lower semi-continuous viscosity solution $u \in LSC(M \times [0, +\infty), \mathbb{R} \cup \{+\infty\})$ of (A.1) in the sense of Barron-Jensen exists, and is unique.*

Proposition A.2. *[27, Theorem 4.1 and Remark 6.3] Let $\varphi(x) \in LSC(M, \mathbb{R} \cup \{\infty\})$, and let u be the Barren-Jensen solution of (A.1). We denote by $\mathcal{C}(x, t, u)$ the set of absolutely continuous curves $\gamma : [0, t] \to M$ with $\gamma(t) = x$ and*

$$
\int_0^t (|L(\gamma(\tau), \dot{\gamma}(\tau), 0)| + |u(\gamma(\tau), \tau)|) d\tau < +\infty.
$$

Fix $(x, t) \in M \times (0, +\infty)$ so that $u(x, t) < +\infty$, then

$$
u(x,t) = \min_{\gamma \in \mathcal{C}(x,t,u)} \left\{ \varphi(\gamma(0)) + \int_0^t L(\gamma(\tau), \dot{\gamma}(\tau), u(\gamma(\tau), \tau)) d\tau \right\},
$$

and the minimum is attained. We define the backward solution semigroup T_t^- as the map $t \mapsto u(\cdot, t)$. Define

$$
F(x, p, u) := H(x, -p, -u)
$$

and let $\bar{T}_t^-$ be the backward solution semigroup corresponding to F, then the forward solution semigroup $T_t^+ \varphi := -\bar{T}_t^-(-\varphi)$ is also well-defined.

Following Fathi [15], one can extend the definition of weak KAM solutions of equation (A.2) by using absolutely continuous calibrated curves instead of C^1 curves.

Definition A.1. *A function $u \in C(M)$ is called a backward (resp. forward) weak KAM solution of (A.2) if*

(1) For each absolutely curve $\gamma : [t', t] \to M$, we have

$$u(\gamma(t)) - u(\gamma(t')) \leq \int_{t'}^{t} L(\gamma(s), \dot{\gamma}(s), u(\gamma(s)))ds.$$

The above condition reads that u is dominated by L and denoted by $u \prec L$.

(2) For each $x \in M$, there exists a absolutely continuous curve $\gamma_- : (-\infty, 0] \to M$ with $\gamma_-(0) = x$ (resp. $\gamma_+ : [0, +\infty) \to M$ with $\gamma_+(0) = x$) such that

$$u(x) - u(\gamma_-(t)) = \int_{t}^{0} L(\gamma_-(s), \dot{\gamma}_-(s), u(\gamma_-(s)))ds, \quad \forall t < 0.$$

$$\text{(resp. } u(\gamma_+(t)) - u(x) = \int_{0}^{t} L(\gamma_+(s), \dot{\gamma}_+(s), u(\gamma_+(s)))ds, \quad \forall t > 0.\text{)}$$

The curves satisfying the above equality are called $(u, L, 0)$-calibrated curves.

The following proposition is well-known in the weak KAM theory, one can refer to [33, Lemma 6.2]. For the existence of the calibrated curves, one can see [27, Lemma 6.6].

Proposition A.3. *The following statements are equivalent: u_- is a fixed point of T_t^-; u_- is a backward weak KAM solution of (A.2); u_- is a viscosity solution of (A.2). Similarly, the following statements are equivalent: v_+ is a fixed point of T_t^+; v_+ is a forward weak KAM solution of (A.2); $-v_+$ is a viscosity solution of*

$$F(x, Du(x), u(x)) = 0. \tag{A.3}$$

Proposition A.4. *[27, Lemma 6.7] Let $\gamma_- : (-\infty, 0] \to M$ be a $(u_-, L, 0)$-calibrated curve, then $T_t^+ u_-(\gamma_-(-t)) = u_-(\gamma_-(-t))$ for each $t > 0$.*

Proposition A.5. *[27, Theorem 6.4] Let u_- be the unique viscosity solution of (A.2), then $T_t^+ u_-$ is nonincreasing in t, and converges to a forward weak KAM solution u_+ of (A.2) uniformly.*

Proposition A.6. *Define the projected Aubry set*

$$\mathcal{I}_{(u_-, u_+)} := \{x \in M : \ u_-(x) = u_+(x)\}.$$

This set is nonempty. We also have $u_- = \lim_{t \to +\infty} T_t^- u_+$. From weak KAM point of view, we call (u_-, u_+) a conjugate pair.

Proof. Since u_- is unique, by Proposition A.5, $T_t^- u_+$ is nondecreasing in t and uniformly converges to the unique backward weak KAM solution u_-. Let $\gamma_- : (-\infty, 0] \to M$ be a $(u_-, L, 0)$-calibrated curve. By Proposition A.4, for each $t > 0$ we have $T_t^+ u_-(\gamma_-(-t)) = u_-(\gamma_-(-t))$. Since M is compact, let $x^* \in M$ such that $d(\gamma_-(-t_n), x^*) \to 0$ as $t_n \to +\infty$. The following inequality holds

$$\begin{aligned}|T_{t_n}^+ u_-(\gamma_-(-t_n)) - u_+(x^*)| \leq & |T_{t_n}^+ u_-(\gamma_-(-t_n)) - u_+(\gamma_-(-t_n))| \\ & + |u_+(\gamma_-(-t_n)) - u_+(x^*)|.\end{aligned}$$

The function u_+ is Lipschitz continuous (see Lemma 2.1). Thus, as $t_n \to +\infty$,

$$|u_+(\gamma_-(-t_n)) - u_+(x^*)| \to 0.$$

Since $T_t^+ u_-$ converges to u_+ uniformly, then

$$|T_{t_n}^+ u_-(\gamma_-(-t_n)) - u_+(\gamma_-(-t_n))| \to 0.$$

Therefore, the limit of $T_{t_n}^+ u_-(\gamma_-(-t_n))$ is $u_+(x^*)$. On the other hand, we have

$$T_{t_n}^+ u_-(\gamma_-(-t_n)) = u_-(\gamma_-(-t_n)),$$

which tends to $u_-(x^*)$ by the continuity of u_-. We conclude that $u_+(x^*) = u_-(x^*)$. □

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