Viscosity solutions of contact Hamilton-Jacobi equations with Hamiltonians depending periodically on unknown functions

Panrui Ni Kaizhi Wang Jun Yan

April 16, 2022

Abstract

Assume $H = H(x, u, p)$ with $(x, p) \in T^*M$ and $u \in \mathbb{S}$, is smooth and satisfies Tonelli conditions in p, Lipschitz continuity condition in u, where M is a compact connected smooth manifold without boundary. We find a compact interval $[c_1, c_2]$ such that equation

$$H(x, u(x), \partial_x u(x)) = c$$

has solutions if and only if $c \in [c_1, c_2]$. We also study the long-time behavior of the unique viscosity solution u^c of

$$\partial_t u(x,t) + H(x, u(x,t), \partial_x u(x,t)) = c, \quad u(x, 0) = \varphi(x) \in C(M, \mathbb{R}).$$

If $c \in [c_1, c_2]$, u^c is bounded by a constant independent of c and Lipschitz with respect to the argument x with a Lipschitz constant independent of c and φ. If $c \notin [c_1, c_2]$, then the long-time average of u^c can be characterized by a function $c \mapsto \rho(c)$ which admits a modulus of continuity. We obtain these results by analyzing properties of a kind of one-parameter semigroups of operators. All the aforementioned results show the fundamental difference between Hamilton-Jacobi equations with Hamiltonians $H(x, u, p)$ and $\bar{H}(x, p)$.

Keywords. Viscosity solutions; existence; long-time behavior; weak KAM theory

Contents

Panrui Ni: School of Mathematical Sciences, Fudan University, Shanghai 200433, China; e-mail: prni18@fudan.edu.cn

Kaizhi Wang (Corresponding author): School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, China; e-mail: kzwang@sjtu.edu.cn

Jun Yan: School of Mathematical Sciences, Fudan University, Shanghai 200433, China; e-mail: yanjun@fudan.edu.cn

Mathematics Subject Classification (2010): 37J55; 35F20; 49L25

1 Introduction and main results

The study of the theory of viscosity solutions [3] of Hamilton-Jacobi equations

$$\partial_t u(x,t) + H(x, u(x,t), \partial_x u(x,t)) = 0 \tag{1.1}$$

and

$$H(x, u(x), \partial_x u(x)) = 0 \tag{1.2}$$

has a long history. See, for instance, [5], [12], [14] and the references therein. We will deal only with viscosity solutions in this paper and thus we mean by "solutions" viscosity solutions. There are numerous results on the existence, uniqueness, stability, and long-time behavior problems for the above first-order partial differential equations, especially for those where $H(x, u, p)$ does not contain the argument u, that is, the corresponding characteristic equations are classical Hamiltonian systems defined on symplectic manifolds.

In view of the relationship between contact Hamiltonian systems and Hamilton-Jacobi equations (1.1) and (1.2), we call (1.1) and (1.2) contact Hamilton-Jacobi equations. The present paper is devoted to the study of the existence and long-time behavior of solutions of equations (1.1) and (1.2), respectively, where the Hamiltonian $H(x, u, p)$ is 1-periodic in the argument u. To the best of our knowledge, little has been known about properties of solutions of these kinds of Hamilton-Jacobi equations, at least from the dynamical point of view. Our tools come from [21, 23, 24], where the authors extended part of Mather and weak KAM theories [17, 6] from classical Hamiltonian systems to contact Hamiltonian systems. An implicit variational principle [21] plays an essential role there. An alternative notable variational formulation was provided in [1, 2] in the light of Herglotz' work [9, 10], which was given in an explicit form with nonholonomic constraints. Using the Herglotz' variational principle, various kinds of representation formulas of solutions of (1.1) were obtained in [11]. See [8, 16, 18] for more on weak KAM type results for the dicounted Hamiltonian system, which is a special kind of contact Hamiltonain systems and has significant physical, optimal control and economics backgrounds.

1.1 Assumptions

Assume M is a compact connected and smooth Riemannian manifold without boundary. Let $H : T^*M \times \mathbb{R} \to \mathbb{R}$ be a C^3 function satisfying:

(H1) (Positive definiteness). For every $(x, u, p) \in T^*M \times \mathbb{R}$, the second partial derivative $\frac{\partial^2 H}{\partial p^2}(x, u, p)$ is positive definite as a quadratic form.

(H2) (Superlinearity). For every $(x, u) \in M \times \mathbb{R}$, $H(x, u, p)$ is superlinear in p.

(H3) (Lipschitz continuity). There exists $\lambda > 0$ such that $|\frac{\partial H}{\partial u}(x, u, p)| \leq \lambda$ for all $(x, u, p) \in T^*M \times \mathbb{R}$.

(H4) (Periodicity). $H(x, u + 1, p) \equiv H(x, u, p)$.

Remark 1.1. *Since results in this paper are based on the variational principle introduced in [21], which was proved under C^3 assumption for a technical reason, we assume $H(x, u, p)$ is of class C^3 here.*

Remark 1.2. *For convenience, we denote $(x, p) \in T^*M$, $u \in \mathbb{R}$, by $(x, u, p) \in T^*M \times \mathbb{R}$. Assumptions (H1), (H2) are classical Tonelli conditions with respect to the argument p. It is clear that if $\bar{H}(x, p)$ is a classical Tonelli Hamiltonian, then the function $(x, u, p) \mapsto \sin(2\pi u) + \bar{H}(x, p)$ satisfies (H1)-(H4).*

Remark 1.3. *Except for Section 2, we always assume (H1)-(H4). In Section 2, we will recall some preliminary results under more general assumptions (H1)-(H3).*

The Lagrangian $L(x, u, \dot{x})$ associated with $H(x, u, p)$ is defined as

$$L(x, u, \dot{x}) := \sup_{p \in T_x^*M} \{\langle \dot{x}, p \rangle_x - H(x, u, p)\},$$

where $\langle \cdot, \cdot \rangle$ represents the canonical pairing between the tangent space and cotangent space. Since $H(x, u, p)$ satisfies (H1)-(H4), then one can deduce that

(L1) (Positive definiteness). For every $(x, u, \dot{x}) \in TM \times \mathbb{R}$, the second partial derivative $\frac{\partial^2 L}{\partial \dot{x}^2}(x, u, \dot{x})$ is positive definite as a quadratic form.

(L2) (Superlinearity). For every $(x, u) \in M \times \mathbb{R}$, $L(x, u, \dot{x})$ is superlinear in $\dot{x}$.

(L3) (Lipschitz continuity). There exists $\lambda > 0$ such that $|\frac{\partial L}{\partial u}(x, u, \dot{x})| \leq \lambda$ for all $(x, u, \dot{x}) \in TM \times \mathbb{R}$.

(L4) (Periodicity). $L(x, u + 1, \dot{x}) \equiv L(x, u, \dot{x})$.

1.2 Main results

Our first main result deals with the existence problem for stationary equations. We characterize the real numbers c for which equation (E) below admits solutions in the following way.

Main Result 1. *There exist c_1, $c_2 \in \mathbb{R}$ with $c_1 \leq c_2$, such that the stationary equation*

$$H(x, u(x), \partial_x u(x)) = c \tag{E}$$

has solutions if and only if $c \in [c_1, c_2]$.

Remark 1.4. *Consider a special case of (E)*

$$\frac{1}{2}(u')^2 - \cos u = c, \quad x \in \mathbb{T}^1, \tag{1.3}$$

which can be derived by the Sine-Laplace equation

$$u'' + \sin u = 0.$$

According to [4, Proposition 3.2], (1.3) admits solutions when $c \in [-1, 1]$.

Remark 1.5. *Recall that for classical Tonelli Hamiltonian $\bar{H}(x,p)$, there is a unique real number c such that the stationary Hamitlon-Jacobi equation*

$$\bar{H}(x, \partial_x u(x)) = c \tag{HJ}$$

has solutions. The number c is called the effective Hamiltonian [13] or Mañé critical value [15]. For $H(x,u,p)$ satisfying (H1)-(H3), it was proved in [23] that there exists a constant c such that equation (E) *has solutions. But the structure of the set $\mathcal{C}$ of all such c's was not discussed there. The set $\mathcal{C}$ can be a singleton, a compact interval, an infinite interval, or even $\mathbb{R}$. This is an essential difference between equation* (E) *and equation* (HJ).

Our approach is dynamical in nature and the analysis of properties of a kind of one-parameter semigroups of operators plays an essential role in the proof of Main Result 1. Wang, Wang and Yan [21] provided a variational principle for contact Hamiltonian systems

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p}(x,u,p), \\ \dot{p} = -\dfrac{\partial H}{\partial x}(x,u,p) - \dfrac{\partial H}{\partial u}(x,u,p)p, \\ \dot{u} = \dfrac{\partial H}{\partial p}(x,u,p)\cdot p - H(x,u,p), \end{cases} \tag{1.4}$$

where the Hamiltonian H satisfies (H1)-(H3). The variational principle can be regarded as a contact counterpart of Tonelli variational principle for classical Hamiltonian systems. Based on the variational principle, they [23] introduced two kinds of Lax-Oleinik type solution semigroups of evolutionary equations (C) below, denoted by $\{T_t^{\pm}\}_{t\geq 0}$. We continue to use $\{T_t^{-}\}_{t\geq 0}$ to study the long-time behavior of solutions of Cauchy problem (C) in the following.

Let us look back at the history of the study of the long-time behavior of solutions of Hamilton-Jacobi equations using the Lax-Oleinik semigroup and dynamical methods. Fathi [6] introduced the Lax-Oleinik semigroups for the classical Hamiltonian system

$$\begin{cases} \dot{x} = \dfrac{\partial \bar{H}}{\partial p}(x,p) \\ \dot{p} = -\dfrac{\partial \bar{H}}{\partial x}(x,p) \end{cases}$$

to establish the weak KAM theory connecting Mather theory [17] and the theory of solutions of Hamilton-Jacobi equations (HJ) and

$$\partial_t u(x,t) + \bar{H}(x, \partial_x u(x,t)) = 0, \tag{1.5}$$

whose solutions can be represented by the Lax-Oleinik semigroup. Fathi showed the convergence of $u(x,t)+ct$ as t approaches infinity by showing the convergence of the Lax-Oleinik semigroup, where $u(x,t)$ is an arbitrary solution of (1.5) and c is the Mañé critical value of $\bar{H}$. After this, lots of interesting work appeared in this direction. See, for example, [12] and references therein for more details. It is worth mentioning that the Lax-Oleinik semigroup may not converge for time-periodic Hamiltonian $\bar{H}(t,x,p)$. The second author of this paper analyzed the long-time behavior of solutions of Hamilton-Jacobi equations with time-periodic Hamiltonian $\bar{H}(t,x,p)$ by introducing the notion of optimal asymptotic bounds [22]. The second and third authors of this paper introduced a new kind of operators with convergence for time-periodic Hamiltonian $\bar{H}(t,x,p)$ in [19, 20]. Using this new kind of operators, one can get all solutions of the corresponding Hamilton-Jacobi equations.

The second main result concerns with the long-time behavior of solutions of evolutionary equations. Let $u^c(x,t)$ denote the unique solution of the Cauchy problem

$$\begin{cases} \partial_t u(x,t) + H(x,u(x,t),\partial_x u(x,t)) = c, \quad (x,t) \in M \times (0,+\infty), \\ u(x,0) = \varphi(x), \quad x \in M. \end{cases} \tag{C}$$

Main Result 2. *Let $\varphi \in C(M,\mathbb{R})$ and $c \in [c_1,c_2]$. Then*

(1) There exists a constant $K_1 > 0$ depending only on the initial data φ and H, such that

$$|u^c(x,t)| \le K_1, \quad \forall x \in M,\ \forall t \ge 0.$$

(2) There exists a constant $K_2 > 0$ depending only on H, such that

$$\operatorname*{ess\,sup}_{x\in M} |\partial_x u^c(x,t)| \le K_2, \quad \forall t > 1.$$

Remark 1.6. *This result guarantees the boundedness of u^c and provides a Lipschitz estimate of u^c with respect to the argument x. When H is strictly increasing in the argument u, u^c converges to the unique solution of* (E) *for all initial data φ [24].*

At last, we consider the long-time behavior of the solution of the evolutionary equation when $c \notin [c_1,c_2]$. We prove the third main result using an interesting connection between the Lax-Oleinik semigroup and the homeomorphisms of the circle. The third main result also shows the essential difference between equations (1.5) and (C).

Main Result 3. *Let $\varphi \in C(M,\mathbb{R})$ and $c \notin [c_1,c_2]$. Then*

(1) The limit $\lim_{t\to+\infty} u^c(x,t)/t =: \rho(c)$ exits and is independent of φ and x.

(2) The function $(x,t) \mapsto |u^c(x,t) - \rho(c)t|$ is bounded on $M \times [1,+\infty)$ by a constant depending only on c.

(3) The function $c \mapsto \rho(c)$ is nondecreasing and continuous with a modulus locally. More precisely, for each compact connected interval $I \subset (-\infty,c_1) \cup (c_2,+\infty)$, and $[c'',c'] \subset I$ with $c'' < c'$, we have

$$0 \le \rho(c') - \rho(c'') \le \omega(c' - c'').$$

Here ω is a nondecreasing real function depending on I, and satisfying $\lim_{r\to 0+} \omega(r) = 0$.

Remark 1.7. *The function $\rho : (-\infty,c_1) \cup (c_2,+\infty) \to \mathbb{R}$ can be extended to the whole real line $\mathbb{R}$. The extended function is still nondecreasing and continuous with a modulus locally. When $c \in [c_1,c_2]$, we have $\rho(c) \equiv 0$ by Main Result 2 (1).*

The rest of the paper is organized as follows. Section 2 gives the basic definitions and preliminaries required for our subsequent work. In Section 3, we show Main Results 1, 2 and 3.

2 Preliminaries

We list notations which will be used later in the present paper.

2.1 Notations

- $\mathrm{diam}(M)$ denotes the diameter of M.
- Denote by d the distance induced by the Riemannian metric g on M.
- Denote by $\|\cdot\|$ the norms induced by g on both tangent and cotangent spaces of M.
- $C(M,\mathbb{R})$ stands for the space of continuous functions on M, $\|\cdot\|_\infty$ denotes the supremum norm on it.
- $\mathrm{ess\,sup}_M |f(x)|$ stands for the essential supremum of $f(x)$ on M.
- For each $t \in \mathbb{R}$, $\{t\} = t \pmod 1$ denotes the fractional part of t and $[t]$ denotes the greatest integer not greater than t.
- Given a, b, δ, $T \in \mathbb{R}$ with $a < b$, $0 < \delta < T$, let

$$\Omega_{a,b,\delta,T} := M \times [a,b] \times M \times [\delta, T].$$

All the results in this section come from [21, 23], and hold true under assumptions (H1)-(H3).

2.2 Variational principle and action functions

Proposition 2.1. ***(Implicit variational principle).*** *For any given $x_0 \in M$ and $u_0 \in \mathbb{R}$, there exists a unique continuous function $h_{x_0,u_0}(x,t)$ defined on $M \times (0,+\infty)$ satisfying*

$$h_{x_0,u_0}(x,t) = u_0 + \inf_{\substack{\gamma(t)=x \\ \gamma(0)=x_0}} \int_0^t L(\gamma(\tau), h_{x_0,u_0}(\gamma(\tau),\tau), \dot\gamma(\tau))d\tau, \tag{2.1}$$

where the infimum is taken among the Lipschitz continuous curves $\gamma : [0,t] \to M$ and can be achieved. Any minimizer is of class C^1. Let γ be a minimizer and

$$x(s) := \gamma(s), \quad u(s) := h_{x_0,u_0}(x(s),s), \quad p(s) := \frac{\partial L}{\partial \dot x}(x(s),u(s),\dot x(s)).$$

Then $(x(s),u(s),p(s))$ satisfies the contact Hamilton's equations (1.4) with $x(0) = x_0$, $x(t) = x$ and $\lim_{s\to 0^-} u(s) = u_0$.

Functions $(x_0,u_0,x,t) \mapsto h_{x_0,u_0}(x,t)$ are called implicit action functions. The following properties of $h_{x_0,u_0}(x,t)$ are very useful in the following.

Proposition 2.2. ***(Monotonicity property I).*** *For any given $x_0 \in M$ and $u_1, u_2 \in \mathbb{R}$, we have*

$$u_1 < u_2 \Rightarrow h_{x_0,u_1}(x,t) < h_{x_0,u_2}(x,t), \quad \forall (x,t) \in M \times (0,+\infty).$$

Proposition 2.3. ***(Monotonicity property II).*** *Given two functions L_1 and L_2 satisfying (L1)-(L3), $x_0 \in M$ and $u_0 \in \mathbb{R}$, if $L_1 < L_2$, then $h^{L_1}_{x_0,u_0}(x,t) < h^{L_2}_{x_0,u_0}(x,t)$ for all $(x,t) \in M \times (0,+\infty)$, where $h^{L_i}_{x_0,u_0}(x,t)$ denotes the implicit action function associated with L_i, $i = 1,2$.*

Proposition 2.4. ***(Markov property).*** *For any given $x_0 \in M$ and $u_0 \in \mathbb{R}$, we have*

$$h_{x_0,u_0}(x,t+s) = \inf_{y\in M} h_{y,h_{x_0,u_0}(y,t)}(x,s), \quad \forall t,\ s>0, \quad \forall x \in M.$$

Moreover, the infimum is attained at y if and only if there is a minimizer γ of $h_{x_0,u_0}(x,t+s)$ such that $\gamma(t) = y$.

Proposition 2.5. ***(Local Lipschitz continuity).*** *Given $a, b, \delta, T \in \mathbb{R}$ with $a < b$ and $0 < \delta < T$, the function $(x_0,u_0,x,t) \mapsto h_{x_0,u_0}(x,t)$ is Lipschitz continuous on $\Omega_{a,b,\delta,T}$.*

For each $c \in \mathbb{R}$, since $L + c$ satisfies all the assumptions imposed on L, then the variational principle and all the results established for L are still correct for $L + c$. Denote by $h^c_{x_0,u_0}(x,t)$ the implicit action function associated with $L + c$.

Proposition 2.6. ***(Monotonicity property III).*** *Given $x_0 \in M$, $u_0 \in \mathbb{R}$ and $c_1, c_2 \in \mathbb{R}$, if $c_1 < c_2$, then $h^{c_1}_{x_0,u_0}(x,t) < h^{c_2}_{x_0,u_0}(x,t)$ for all $(x,t) \in M \times (0,+\infty)$.*

Proposition 2.7. *Given $a, b, \delta, T \in \mathbb{R}$ with $a < b$ and $0 < \delta < T$, for any $(x_0,u_0,x,t) \in \Omega_{a,b,\delta,T}$ and $c_1,\ c_2 \in \mathbb{R}$, we have*

$$|h^{c_1}_{x_0,u_0}(x,t) - h^{c_2}_{x_0,u_0}(x,t)| \le e^{\lambda t} t|c_1 - c_2| \le e^{\lambda T} T|c_1 - c_2|,$$

where λ is as in (L3).

2.3 Solution semigroups

The authors of [23] introduced two kinds of solution semigroups, denoted by $\{T^-_t\}_{t\ge 0}$ and $\{T^+_t\}_{t\ge 0}$, which are called backward solution semigroup and forward solution semigroup, respectively. In this paper, since we will only use $\{T^-_t\}_{t\ge 0}$, we denote $\{T^-_t\}_{t\ge 0}$ by $\{T_t\}_{t\ge 0}$ for brevity in the following.

Proposition 2.8. ***(Solution semigroup).*** *There is a unique semigroup of operators $\{T_t\}_{t\ge 0} : C(M,\mathbb{R})^{\circlearrowleft}$ such that*

$$T_t\varphi(x) = \inf_{\gamma(t)=x} \left\{ \varphi(\gamma(0)) + \int_0^t L(\gamma(\tau), T_\tau\varphi(\gamma(\tau)), \dot\gamma(\tau)) d\tau \right\},$$

where the infimum is taken among the Lipschitz continuous curves $\gamma : [0,t] \to M$ with $\gamma(t) = x$ and can be achieved. For each $\varphi \in C(M,\mathbb{R})$, the function $(x,t) \mapsto T_t\varphi(x)$ is the unique solution of $\partial_t u(x,t) + H(x,u(x,t),\partial_x u(x,t)) = 0$ with $u(x,0) = \varphi(x)$. Furthermore,

$$T_t\varphi(x) = \inf_{y\in M} h_{y,\varphi(y)}(x,t), \quad \forall (x,t) \in M \times [0,+\infty),$$

where h is the implicit action function obtained in Proposition 2.1.

Proposition 2.9. *Given $\varphi, \psi \in C(M,\mathbb{R})$, we have*

(1) if $\varphi < \psi$, then $T_t\varphi < T_t\psi$ for all $t \ge 0$.

(2) the function $(x,t) \mapsto T_t\varphi(x)$ is locally Lipschitz on $M \times (0,+\infty)$.

Denote by $T_t^c\varphi(x)$ the solution semigroup associated with $L + c$.

Proposition 2.10. *Given $\varphi \in C(M,\mathbb{R})$, let*

$$c_1 := \sup\{c \mid \inf_{(x,t)\in M\times[0,+\infty)} T_t^c\varphi(x) = -\infty\}, \quad c_2 := \inf\{c \mid \sup_{(x,t)\in M\times[0,+\infty)} T_t^c\varphi(x) = +\infty\}.$$

Then $-\infty \leq c_1 < +\infty$ and $-\infty < c_2 \leq +\infty$. Moreover, $c_1 \leq c_2$.

3 Proofs of Main Results

Before giving the proofs of Main Results 1, 2, 3, we show a Lipschitz estimate for implicit action functions first.

3.1 Lipschitz estimate

Lemma 3.1. *For any given $c \in \mathbb{R}$, $n \in \mathbb{Z}$, and $\varphi \in C(M,\mathbb{R})$,*

$$T_t^c(\varphi + n)(x) = T_t^c\varphi(x) + n, \quad \forall (x,t) \in M \times [0,+\infty).$$

Proof. By definition, we have

$$T_t^c\varphi(x) + n = \inf_{\gamma(t)=x}\left\{\varphi(\gamma(0)) + n + \int_0^t L(\gamma(\tau), T_\tau^c\varphi(\gamma(\tau)), \dot\gamma(\tau))\mathrm{d}\tau + ct\right\},$$

where the infimum is taken among the Lipschitz continuous curves $\gamma : [0,t] \to M$ with $\gamma(t) = x$. In view of (L4) and Proposition 2.8, we get that

$$T_t^c\varphi(x) + n = \inf_{\gamma(t)=x}\left\{\varphi(\gamma(0)) + n + \int_0^t L(\gamma(\tau), T_\tau^c\varphi(\gamma(\tau)) + n, \dot\gamma(\tau))\mathrm{d}\tau + ct\right\} = T_t^c(\varphi + n)(x).$$

□

For any given $\varphi \in C(M,\mathbb{R})$, recall that

$$c_1 := \sup\{c \mid \inf_{(x,t)\in M\times[0,+\infty)} T_t^c\varphi(x) = -\infty\}, \quad c_2 := \inf\{c \mid \sup_{(x,t)\in M\times[0,+\infty)} T_t^c\varphi(x) = +\infty\}, \tag{3.1}$$

and $-\infty \leq c_1 < +\infty$, $-\infty < c_2 \leq +\infty$, $c_1 \leq c_2$.

Lemma 3.2. *Let $\varphi \in C(M,\mathbb{R})$. Both c_1 and c_2 are real numbers. The function $(x,t) \mapsto T_t^c\varphi(x)$ is bounded on $M \times [0,+\infty)$ if c belongs to the finite interval (c_1, c_2).*

Proof. We prove the first assertion first. If we can find a real number c' such that

$$\sup_{M\times[0,+\infty)} T_t^{c'}\varphi(x) = +\infty, \tag{3.2}$$

then by the definition of c_2, one can deduce that $c_2 \in \mathbb{R}$. Let c'' denote an arbitrary real number, if $c_1 = -\infty$; $c'' = c_1 + 1$, if $c_1 \in \mathbb{R}$. Then by the definition of c_1, it is clear that $\inf_{M\times[0,+\infty)} T_t^{c''}\varphi(x) > -\infty$.

Let $c' = \lambda + c'' + 1$. Let $\gamma:\ [0,t] \to M$ be a minimizer of $T_t^{c'}\varphi(x)$ with $\gamma(t) = x$. We have

$$\begin{aligned} T_t^{c'}\varphi(x) - T_t^{c''}\varphi(x) &\geq \varphi(\gamma(0)) + \int_0^t L(\gamma(\tau), T_\tau^{c'}\varphi(\gamma(\tau)), \dot\gamma(\tau))d\tau + c't \\ &\quad - \varphi(\gamma(0)) - \int_0^t L(\gamma(\tau), T_\tau^{c''}\varphi(\gamma(\tau)), \dot\gamma(\tau))d\tau - c''t \\ &\geq -\lambda \int_0^t |T_t^{c'}\varphi(\gamma(\tau)) - T_t^{c''}\varphi(\gamma(\tau))| (\text{mod } 1) d\tau + (c' - c'')t \\ &\geq t, \end{aligned}$$

implying (3.2) holds. Thus, $c_2 \in \mathbb{R}$. Since the proof of $c_1 \in \mathbb{R}$ is quite similar to the one of $c_2 \in \mathbb{R}$, we omit it for brevity.

The second assertion is a direct consequence of the definitions of c_1, c_2 and the first assertion. □

Lemma 3.3. *Both c_1 and c_2 depend only on H.*

Proof. We aim to prove that the values of c_1 and c_2 do not depend on the initial data φ. Fix $\varphi_0 \in C(M,\mathbb{R})$, let c_1 and c_2 be defined as in (3.1) with $\varphi = \varphi_0$. Given any $\phi \in C(M,\mathbb{R})$, there exist $n_1, n_2 \in \mathbb{Z}$ such that

$$\varphi_0 + n_1 \leq \phi \leq \varphi_0 + n_2.$$

By Proposition 2.9 and Lemma 3.1, we get that

$$T_t^c\varphi_0 + n_1 \leq T_t^c\phi \leq T_t^c\varphi_0 + n_2, \quad \forall c \in \mathbb{R},\ \forall t \geq 0. \tag{3.3}$$

If $c > c_2$, then $\sup_{M\times[0,+\infty)} T_t^c\varphi_0(x) + n_1 = +\infty$. By the first inequality in (3.3), we get

$$\sup_{M\times[0,+\infty)} T_t^c\phi(x) = +\infty.$$

If $c < c_2$, then $\sup_{M\times[0,+\infty)} T_t^c\varphi_0(x) + n_2 < +\infty$. By the second inequality in (3.3), we have $\sup_{M\times[0,+\infty)} T_t^c\phi(x) < +\infty$. So, we deduce that $c_2 = \inf\{c \mid \sup_{(x,t)\in M\times[0,+\infty)} T_t^c\phi(x) = +\infty\}$, which means c_2 is independent of the initial data ϕ. The assertion for c_1 can be obtained in a similar manner.

□

Given a, b, δ, $T \in \mathbb{R}$ with $a < b$ and $0 < \delta < T$, recall that

$$\Omega_{a,b,\delta,T} := M \times [a,b] \times M \times [\delta, T].$$

Lemma 3.4. *Let c_1 and c_2 be defined as in* (3.1)*. There is a constant $C_{a,b,\delta,T} > 0$, such that*

$$|h^c_{x_0,u_0}(x,t)| \leq C_{a,b,\delta,T}, \quad \forall (x_0,u_0,x,t) \in \Omega_{a,b,\delta,T}, \quad \forall c \in (c_1,c_2),$$

where $C_{a,b,\delta,T}$ depends only on a, b, δ and T.

Proof. Let

$$k = \frac{\text{diam(M)}}{\delta}, \quad A = \sup_{\|\dot{x}\| \le k} L(x, 0, \dot{x}), \quad B = \inf_{(x,\dot{x}) \in TM} L(x, 0, \dot{x}).$$

Boundedness from below. Given any $(x_0, u_0, x, t) \in \Omega_{a,b,\delta,T}$, let $\gamma : [0, t] \to M$ be a minimizer of $h^c_{x_0,u_0}(x, t)$ and $u^c(s) = h^c_{x_0,u_0}(\gamma(s), s)$, $s \in [0, t]$. Then $u^c(t) = h^c_{x_0,u_0}(x, t)$. We need to show that $u^c(t)$ is bounded below by a constant which depends only on a, b, δ and T, There are three possibilities:

(i) $u^c(t) > 0$. It is clear that $u^c(t)$ is bounded below by 0;

(ii) $u^c(s) < 0, \forall s \in [0, t]$;

(iii) there exists $s_0 \in [0, t]$ such that $u^c(s_0) = 0$ and $u^c(s) \le 0, \forall s \in [s_0, t]$.

Case (ii): Note that u^c satisfies

$$\dot{u}^c(s) = L(\gamma(s), u^c(s), \dot{\gamma}(s)) + c \ge L(\gamma(s), 0, \dot{\gamma}(s)) + \lambda u^c(s) + c \ge B + \lambda u^c(s) + c_1, \quad \forall s \in [0, t]$$

and $u^c(0) = u_0 \in [a, b]$. Consider the solution $w_1(s)$ of the following Cauchy problem

$$\dot{w}_1(s) = B + \lambda w_1(s) + c_1, \quad w_1(0) = u_0.$$

It is easy to see that $w_1(s) = u_0 e^{\lambda s} + \frac{B+c_1}{\lambda}(e^{\lambda s} - 1)$. Using the comparison theorem of solutions of ordinary differential equations, we have

$$u^c(t) \ge w_1(t) = u_0 e^{\lambda t} + \frac{B + c_1}{\lambda}(e^{\lambda t} - 1) \ge -|a| e^{\lambda T} - \frac{|B + c_1|}{\lambda}(e^{\lambda T} - 1).$$

Case (iii): In this case, $\dot{u}^c(s) \ge B + \lambda u^c(s) + c_1$ for $s \in [s_0, t]$ and $u^c(s_0) = 0$. Let $w_2(s)$ be the solution of the following Cauchy problem

$$\dot{w}_2(s) = B + \lambda w_2(s) + c_1, \quad w_2(s_0) = 0.$$

Then $w_2(s) = \frac{B+c_1}{\lambda}\left(e^{\lambda(s-s_0)} - 1\right)$. Thus, we have

$$u^c(t) \ge w_2(t) = \frac{B + c_1}{\lambda}\left(e^{\lambda(t-s_0)} - 1\right) \ge -\frac{|B + c_1|}{\lambda}\left(e^{\lambda T} - 1\right).$$

Therefore, we get

$$h^c_{x_0,u_0}(x, t) \ge -|a| e^{\lambda T} - \frac{|B + c_1|}{\lambda}(e^{\lambda T} - 1).$$

Boundedness from above. Civen any $(x_0, u_0, x, t) \in \Omega_{a,b,\delta,T}$, let $\alpha : [0, t] \to M$ be a geodesic between x_0 and x with $\|\dot{\alpha}\| = d(x_0, x)/t \le \text{diam}(M)/\delta = k$. Let $v^c(s) = h^c_{x_0,u_0}(\alpha(s), s)$, $s \in [0, t]$. Then $v^c(t) = h^c_{x_0,u_0}(x, t)$ and $v^c(0) = u_0$. Note that

$$v^c(s_2) - v^c(s_1) \le \int_{s_1}^{s_2} \big(L(\alpha(s), v^c(s), \dot{\alpha}(s)) + c\big) ds, \quad 0 \le s_1 \le s_2 \le t.$$

Thus, we get

$$\dot{v}^c(s) \le L(\alpha(s), v^c(s), \dot{\alpha}(s)) + c \le L(\alpha(s), 0, \dot{\alpha}(s)) + \lambda |v^c(s)| + c_2.$$

We need to show that $v^c(t)$ is bounded from above by a constant which depends only on a, b, δ and T. There are three possibilities:

(1) $v^c(t) < 0$. In this case, $v^c(t)$ is bounded from above by 0;

(2) $v^c(s) > 0, \forall s \in [0,t]$;

(3) there exists $s' \in [0,t]$ such that $v^c(s') = 0$ and $v^c(s) \geq 0, \forall s \in [s',t]$.

Case (2): Since $v^c > 0$ for all $s \in [0,t]$, then

$$\dot{v}^c(s) \leq L(\alpha(s), 0, \dot{\alpha}(s)) + \lambda|v^c(s)| + c_2 \leq A + \lambda v^c(s) + c_2,$$

and $v^c(0) = u_0$. Let $w_3(s)$ be the solution of the following Cauchy problem

$$\dot{w}_3(s) = A + \lambda w_3(s) + c_2, \quad w_3(0) = u_0.$$

One can easily obtain that $w_3(s) = u_0 e^{\lambda s} + \frac{A+c_2}{\lambda}(e^{\lambda s} - 1)$. Thus, we get

$$v^c(t) \leq w_3(t) = u_0 e^{\lambda t} + \frac{A+c_2}{\lambda}(e^{\lambda t} - 1) \leq |b|e^{\lambda T} + \frac{|A+c_2|}{\lambda}(e^{\lambda T} - 1).$$

Case (3): In this case $\dot{v}^c(s) \leq A + \lambda v^c(s) + c_2$, for $s \in [s',t]$ and $v^c(s') = 0$. Let $w_4(s)$ be the solution of the following Cauchy problem

$$\dot{w}_4(s) = A + \lambda w_4(s) + c_2, \quad w_4(s') = 0.$$

Then $w_4(s) = \frac{A+c_2}{\lambda}\left(e^{\lambda(s-s')} - 1\right)$. Using the comparison theorem of solutions of ordinary differential equations, we have

$$v^c(t) \leq w_4(t) = \frac{A+c_2}{\lambda}\left(e^{\lambda(t-s')} - 1\right) \leq \frac{|A+c_2|}{\lambda}\left(e^{\lambda T} - 1\right).$$

Hence, we have

$$h^c_{x_0,u_0}(x,t) \leq |b|e^{\lambda T} + \frac{|A+c_2|}{\lambda}(e^{\lambda T} - 1).$$

□

Lemma 3.5. *Let c_1 and c_2 be defined as in* (3.1)*. There is a constant $K_{a,b,\delta,T} > 0$ such that for any $(x_0, u_0, x, t) \in \Omega_{a,b,\delta,T}$, any $c \in (c_1, c_2)$, any minimizer γ of $h^c_{x_0,u_0}(x,t)$, there holds*

$$|h^c_{x_0,u_0}(\gamma(s), s)| \leq K_{a,b,\delta,T}, \quad \forall s \in [0,t],$$

where $K_{a,b,\delta,T}$ depends only on a, b, δ and T.

Proof. Boundedness from below. By similar arguments used in the first part of the proof of Lemma 3.4, one can show that $h^c_{x_0,u_0}(\gamma(s), s)$ is bounded from below by a constant which depends only on a and T. We omit the details for brevity.

Boundedness from above. We only need to show that there exists a constant $K_{a,b,\delta,T} > 0$ independent of c such that

$$h^c_{x_0,u_0}(\gamma(s), s) \leq K_{a,b,\delta,T}, \quad \forall s \in [0,t].$$

Let $u^c(s) = h^c_{x_0,u_0}(\gamma(s), s)$, $s \in [0,t]$ and $u^c_e = h^c_{x_0,u_0}(x,t)$. Let $C_{a,b,\delta,T}$ be as in the last Lemma. Then $|u^c_e| \leq C_{a,b,\delta,T}$ and there are two possibilities:

(1) $u_e^c > 0$;

(2) $u_e^c \le 0$.

Case (1): We assert that

$$u^c(s) \le \frac{|B+c_1|}{\lambda} + \left(C_{a,b,\delta,T} + 1 + \frac{|B+c_1|}{\lambda}\right) e^{\lambda T}, \quad \forall s \in [0,t].$$

Otherwise, there would be $s_1 \in [0,t]$ such that

$$u^c(s_1) > \frac{|B+c_1|}{\lambda} + \left(C_{a,b,\delta,T} + 1 + \frac{|B+c_1|}{\lambda}\right) e^{\lambda T}.$$

Then there is $s_2 \in [0,t]$ such that $u^c(s_2) = u_e^c$ and

$$u^c(s) > u_e^c > 0, \quad \forall s \in [s_1, s_2].$$

Note that for any $s \in [s_1, s_2]$,

$$\dot{u}^c(s) = L(\gamma(s), u^c(s), \dot{\gamma}(s)) + c \ge L(\gamma(s), 0, \dot{\gamma}(s)) - \lambda |u^c(s)| + c_1 \ge B - \lambda u^c(s) + c_1.$$

Let $w(s)$ be the solution of the following Cauchy problem

$$\dot{w}(s) = B - \lambda w(s) + c_1, \quad w(s_1) = u^c(s_1).$$

Then $w(s) = e^{-\lambda(s-s_1)}\left(u^c(s_1) - \frac{B+c_1}{\lambda}\right) + \frac{B+c_1}{\lambda}$. Thus, we get

$$u^c(s_2) \ge w(s_2) = e^{-\lambda(s_2-s_1)}\left(u^c(s_1) - \frac{B+c_1}{\lambda}\right) + \frac{B+c_1}{\lambda},$$

which together with $u^c(s_1) > \frac{|B+c_1|}{\lambda} + \left(C_{a,b,\delta,T} + 1 + \frac{|B+c_1|}{\lambda}\right) e^{\lambda T}$ implies

$$u^c(s_2) > u_e^c + 1.$$

a contradiction. Hence, the assertion is true.

Case (2): In this case, we assert that

$$u^c(s) \le \frac{|B+c_1|}{\lambda} + \left(2 + \frac{|B+c_1|}{\lambda}\right) e^{\lambda T}, \quad \forall s \in [0,t].$$

If the assertion is not true, there would be $s_1, s_2 \in [0,t]$ such that

$$u^c(s_1) > \frac{|B+c_1|}{\lambda} + \left(2 + \frac{|B+c_1|}{\lambda}\right) e^{\lambda T}, \quad u^c(s_2) = 1,$$

and

$$u^c(s) \ge 1, \quad \forall s \in [s_1, s_2].$$

Note that

$$\dot{u}^c(s) \ge B - \lambda u^c(s) + c_1, \quad \forall s \in [s_1, s_2].$$

Let $v(s)$ be the solution of the following Cauchy problem

$$\dot{v}(s) = B - \lambda v(s) + c_1, \quad v(s_1) = u^c(s_1).$$

Then $v(s) = e^{-\lambda(s-s_1)}\left(u^c(s_1) - \frac{B+c_1}{\lambda}\right) + \frac{B+c_1}{\lambda}$. Thus, in view of $u^c(s_1) > \frac{|B+c_1|}{\lambda} + \left(2 + \frac{|B+c_1|}{\lambda}\right)e^{\lambda T}$ and $u(s_2) = 1$, we have

$$u^c(s_2) > v(s_2) > 1,$$

a contradiction. □

Lemma 3.6. *Let c_1 and c_2 be as defined in* (3.1). *Let $(x_0, u_0) \in M \times [a, b]$. For any $c \in (c_1, c_2)$, the function $(x, t) \mapsto h^c_{x_0,u_0}(x, t)$ is Lipschitz on $M \times [\delta, T]$, and the Lipschitz constant is independent of c.*

Proof. Let γ be a minimizer of $h^c_{x_0,u_0}(x, t)$ and $u^c(s) = h^c_{x_0,u_0}(\gamma(s), s)$, $s \in [0, t]$. According to Lemma 3.5

$$|h^c_{x_0,u_0}(\gamma(s), s)| \leq K_{a,b,\delta,T}, \quad \forall s \in [0, t].$$

Then from (L2) there is a constant $D := D_{a,b,\delta,T} \in \mathbb{R}$ such that

$$L(\gamma(s), u^c(s), \dot{\gamma}(s)) + c \geq \|\dot{\gamma}(s)\| + D + c_1, \quad \forall s \in [0, t].$$

Choose $Q := Q_{a,b,\delta,T} > 0$ such that

$$a + Q\delta - |D + c_1|T > K_{a,b,\delta,T}.$$

We assert that there is $s_0 \in [0, t]$ such that $\|\dot{\gamma}(s_0)\| \leq Q$. If the assertion is not true, then $\|\dot{\gamma}(s)\| > Q$, $\forall s \in [0, t]$. Since

$$\dot{u}^c(s) = L(\gamma(s), u^c(s), \dot{\gamma}(s)) + c \geq \|\dot{\gamma}(s)\| + D + c_1,$$

then

$$\int_0^t \dot{u}^c(s)ds \geq \int_0^t (\|\dot{\gamma}(s)\| + D + c_1)ds.$$

Thus, we get

$$u^c(t) \geq u_0 + Qt + Dt + c_1 t \geq a + Qt + Dt + c_1 t > a + Q\delta - |D + c_1|T > K_{a,b,\delta,T},$$

a contradiction.

Thus, there is $s_0 \in [0, t]$ such that the bound of $\dot{\gamma}(s_0)$ is independent of c. Note that

$$\frac{dH}{ds}(\gamma(s), u^c(s), p(s)) = -\big(H(\gamma(s), u^c(s), p(s)) - c\big)\frac{\partial H}{\partial u}(\gamma(s), u^c(s), p(s)),$$

where $c_1 < c < c_2$. Let $c_0 = \max\{|c_1|, |c_2|\}$, by (H3) we get

$$|H(\gamma(s), u^c(s), p(s))| \leq (|H(\gamma(s_0), u^c(s_0), p(s_0))| + c_0)\, e^{\lambda T} - c_0.$$

Then by (H2), we obtain that the bounds of $\|p(s)\|$ and $\|\dot{\gamma}(s)\|$ are independent of c, depending only on a, b, δ and T.

(i) We first consider the Lipschitz property of $h^c_{x_0,u_0}(x,t)$ with respect to x. Let $\gamma(t)$ be a minimizer of $h^c_{x_0,u_0}(x,t)$ and $\Delta t = d(x,y)$. Then

$$h^c_{x_0,u_0}(y,t) - h^c_{x_0,u_0}(x,t) = h^c_{x_0,u_0}(y,t) - h^c_{x_0,u_0}(\gamma(t-\Delta t), t-\Delta t) + h^c_{x_0,u_0}(\gamma(t-\Delta t), t-\Delta t) - h^c_{x_0,u_0}(x,t).$$

Let $A := h^c_{x_0,u_0}(y,t) - h^c_{x_0,u_0}(\gamma(t-\Delta t), t-\Delta t)$ and $B := h^c_{x_0,u_0}(\gamma(t-\Delta t), t-\Delta t) - h^c_{x_0,u_0}(x,t)$. Let $\alpha : [0,\Delta t] \to M$ be a geodesic with constant speed connecting $\gamma(t-\Delta t)$ and y. Then

$$\|\dot{\alpha}\| = \frac{d(\gamma(t-\Delta t), y)}{d(x,y)} \leq \frac{d(\gamma(t-\Delta t), x) + d(x,y)}{d(x,y)} = 1 + \frac{d(\gamma(t-\Delta t), x)}{d(x,y)}.$$

We will use J_i, $i = 1,2,3,4$ to denote positive constants independent of c in the following. From $d(\gamma(t-\Delta t), x) \leq \int_{t-\Delta t}^t \|\dot{\gamma}(s)\| ds$, we deduce $d(\gamma(t-\Delta t), x) \leq J_1 \Delta t$, since we have proved that $\|\dot{\gamma}\|$ is bounded by a constant independent of c. Thus, $\|\dot{\alpha}(s)\|$ is bounded by a constant independent of c. Hence

$$\begin{aligned} A &\leq \int_{t-\Delta t}^t L(\alpha(s), u^c(\alpha(s), s), \dot{\alpha}(s)) ds \leq J_2 d(x,y), \\ B &= -\int_{t-\Delta t}^t L(\gamma(s), u^c(\gamma(s), s), \dot{\gamma}(s)) ds \leq J_3 d(x,y). \end{aligned}$$

Combining the above two inequalities, we have $h^c_{x_0,u_0}(y,t) - h^c_{x_0,u_0}(x,t) \leq J_4 d(x,y)$. By exchanging the roles of x and y, we get $|h^c_{x_0,u_0}(y,t) - h^c_{x_0,u_0}(x,t)| \leq D_1 d(x,y)$, where D_1 is independent of c.

(ii) Next we prove the Lipschitz property of $h^c_{x_0,u_0}(x,t)$ with respect to t. Let $\gamma(t)$ be a minimizer of $h^c_{x_0,u_0}(x,t)$. Then we have

$$\begin{aligned} h^c_{x_0,u_0}(x,t) - h^c_{x_0,u_0}(x,s) &= h^c_{x_0,u_0}(\gamma(s), s) - h^c_{x_0,u_0}(x,s) + \int_s^t L(\gamma(\tau), u^c(\gamma(\tau), \tau), \dot{\gamma}(\tau)) d\tau \\ &\leq h^c_{x_0,u_0}(\gamma(s), s) - h^c_{x_0,u_0}(x,s) + J_5(t-s). \end{aligned}$$

From (i) we have

$$|h^c_{x_0,u_0}(\gamma(s), s) - h^c_{x_0,u_0}(x,s)| \leq D_1 d(\gamma(s), x) \leq D_1 \int_s^t \|\dot{\gamma}(\tau)\| d\tau \leq J_6(t-s).$$

Here, J_5, J_6 are positive constants independent of c. Therefore, we get

$$|h^c_{x_0,u_0}(x,t) - h^c_{x_0,u_0}(x,s)| \leq D_2 |t-s|,$$

where D_2 is independent of c. □

By slight modification of the proof of Lemma 3.6, one can prove

Corollary 3.1. *Let $(x_0, u_0) \in M \times [a,b]$. For any $c \in (p_1, p_2)$, the function $(x,t) \mapsto h^c_{x_0,u_0}(x,t)$ is Lipschitz on $M \times [\delta, T]$, and the Lipschitz constant is independent of c. More precisely, the Lipschitz constant depends on a, b, δ, T and p_1, p_2.*

3.2 Proof of Main Result 1

For $c \notin [c_1, c_2]$, since a function u is a solution of

$$H(x, u, \partial_x u) = c \tag{3.4}$$

if and only if u is a fixed point of $\{T_t^c\}_{t\geq 0}$, then by the definitions of c_1 and c_2, equation (3.4) has no solutions.

For $c \in (c_1, c_2)$, in view of [23, Step 2 in the proof of Theorem 1.2], $\{T_t^c\varphi(x)\}_{t\geq 1}$ is uniformly bounded and equi-Lipschitz on M, and

$$\varphi_\infty^c(x) := \liminf_{t\to+\infty} \varphi_\infty^c(x)$$

is a solution of (3.4). Note that H is 1-periodic in u and satisfies superlinear growth condition. From Lemma 3.3, c_1, c_2 depend only on H. Hence, $\operatorname{ess\,sup}_M |\partial_x \varphi_\infty^c(x)|$ is bounded by a constant independent of c. Fix $x_0 \in M$, let

$$\tilde{\varphi}_\infty^c(x) := \varphi_\infty^c(x) - [\varphi_\infty^c(x_0)].$$

Then $\tilde{\varphi}_\infty^c$ is still a solution of (3.4). Since $\operatorname{ess\,sup}_M |\partial_x \tilde{\varphi}_\infty^c(x)|$ is bounded by a constant independent of c and $c \in (c_1, c_2)$, then $\tilde{\varphi}_\infty^c$ is bounded by a constant independent of c. By Ascoli Lemma, there are $\{c_n\} \subset (c_1, c_2)$ and $\tilde{\varphi}_\infty^{c_n}(x) \in C(M, \mathbb{R})$ such that $c_2 = \lim_{n\to+\infty} c_n$ and the uniform limit

$$u^*(x) := \lim_{n\to+\infty} \tilde{\varphi}_\infty^{c_n}(x)$$

exists. From the stability property of solutions, it is clear that u^* is a solution of $H(x, u, \partial_x u) = c_2$. By similar arguments we can show that $H(x, u, \partial_x u) = c_1$ also admits solutions.

3.3 Proof of Main Result 2

(1) Let u_i be an arbitrary solution of

$$H(x, u, \partial_x u) = c_i, \quad i = 1, 2.$$

For any $\varphi \in C(M, \mathbb{R})$, there are $N_i^\varphi \in \mathbb{N}$ such that

$$u_i - N_i^\varphi \leq \varphi \leq u_i + N_i^\varphi.$$

Thus, we get

$$u_i - N_i^\varphi \leq T_t^{c_i}\varphi \leq u_i + N_i^\varphi, \quad \forall t > 0.$$

Therefore, for any $c \in [c_1, c_2]$, we have

$$T_t^{c_1}\varphi \leq T_t^c\varphi \leq T_t^{c_2}\varphi, \quad \forall t > 0,$$

which completes the proof of the first assertion.

(2) For any $c \in [c_1, c_2]$, since $h^c_{x_0,u_0+1}(x, 1) = 1 + h^c_{x_0,u_0}(x, 1)$, then

$$|T_t^c\varphi(x) - T_t^c\varphi(y)| \leq \sup_{z\in M} |h^c_{z,T_{t-1}^c\varphi(z)(\text{mod } 1)}(x, 1) - h^c_{z,T_{t-1}^c\varphi(z)(\text{mod } 1)}(y, 1)| \leq l_1 d(x, y), \ \forall t > 1,$$

where the Lipschitz constant l_1 independent of c comes from Lemma 3.6. The proof is complete.

3.4 Proof of Main Result 3

We will prove the three results in Main Result 3 only for the case $c > c_2$. By similar arguments, one can get the proof for the case $c < c_1$.

(1) Since $1 + h_{x_0,u_0}(x, 1) = h_{x_0,u_0+1}(x, 1)$, by Proposition 2.5 we have

$$\begin{aligned} |T^c_t\varphi(x) - T^c_t\varphi(y)| &\leq \sup_{z\in M} |h^c_{z,T^c_{t-1}\varphi(z)}(x, 1) - h^c_{z,T^c_{t-1}\varphi(z)}(y, 1)| \\ &= \sup_{z\in M} |h^c_{z,T^c_{t-1}\varphi(z)(\text{mod } 1)}(x, 1) - h^c_{z,T^c_{t-1}\varphi(z)(\text{mod } 1)}(y, 1)| \\ &\leq l^c_1 d(x, y), \quad \forall t \geq 1, \end{aligned} \tag{3.5}$$

where l^c_1 is the Lipschitz constant of $x \mapsto h^c_{x_0,u_0}(x, 1)$, depending on c. For any given $c > c_2$, the family of continuous functions $\{T^c_t\varphi(x)\}_{t\geq1}$ is equi-Lipschitz.

We denote by $\text{Lip}(l^c_1) \subset C(M, \mathbb{R})$ the set of Lipschitz continuous functions with Lipschitz constant l^c_1. By (3.5), T^c_1 is an operator from $\text{Lip}(l^c_1)$ to itself. For any φ_1, $\varphi_2 \in \text{Lip}(l^c_1)$, from Proposition 2.8 there is $z_2 \in M$ such that

$$T^c_1\varphi_1(x) - T^c_1\varphi_2(x) \leq h^c_{z_2,\varphi_1(z_2)}(x, 1) - h^c_{z_2,\varphi_2(z_2)}(x, 1) \leq l^c_{u_0}\|\varphi_1 - \varphi_2\|_\infty,$$

where $l^c_{u_0}$ is the Lipschitz constant of the function $u_0 \mapsto h^c_{x_0,u_0}(x, 1)$ on $[-A, A]$ and $A := \max\{\|\varphi_1\|_\infty, \|\varphi_2\|_\infty\}$. By changing the roles of φ_1 and φ_2, it is clear that the map $\varphi \mapsto T^c_1\varphi$ is continuous. Thus, for each $m \in \mathbb{N}$ and $x \in M$, we can define

$$\alpha_m(x) = \inf_{\varphi\in\text{Lip}(l^c_1)} (T^c_m\varphi(x) - \varphi(x)), \quad \beta_m(x) = \sup_{\varphi\in\text{Lip}(l^c_1)} (T^c_m\varphi(x) - \varphi(x)).$$

We assert that $\alpha_m(x)$ and $\beta_m(x)$ are well-defined. In fact, since the operator $T^c_1 - id$ has $\mathbb{Z}$-translation invariance, we can choose $\varphi \in \text{Lip}(l^c_1)$ satisfying $\varphi(x_0) \in [0, 1)$, for some $x_0 \in M$. Then $\|\varphi\|_\infty \leq 1 + l^c_1\text{diam}(M)$. Denote the set of such functions by $\mathcal{B}^c_{x_0}$. This set of functions is uniformly bounded and equi-Lipschitz. So $\mathcal{B}^c_{x_0}$ is a compact subset of $C(M, \mathbb{R})$.

Fix $x_0 \in M$, for any $\varphi_1, \varphi_2 \in \mathcal{B}^c_{x_0}$, we may assume that $\varphi_1(x_0) \leq \varphi_2(x_0) < \varphi_1(x_0) + 1$. Then

$$\varphi_1(x) - 2l^c_1\text{diam}(M) \leq \varphi_2(x) \leq \varphi_1(x) + 1 + 2l^c_1\text{diam}(M), \quad \forall x \in M.$$

We can take $N^c \in \mathbb{Z}$ large enough (for example, $N^c = [2l^c_1\text{diam}(M)] + 1$) such that

$$\varphi_1 - N^c \leq \varphi_2 \leq \varphi_1 + 1 + N^c.$$

Note that N^c depends only on c. For any $m \in \mathbb{N}$, we get

$$T^c_m\varphi_1 - N^c \leq T^c_m\varphi_2 \leq T^c_m\varphi_1 + 1 + N^c.$$

Then

$$T^c_m\varphi_1 - N^c - (\varphi_1 + 1 + N^c) \leq T^c_m\varphi_2 - \varphi_2 \leq T^c_m\varphi_1 + 1 + N^c - (\varphi_1 - N^c),$$

which implies

$$(T^c_m\varphi_1 - \varphi_1) - (2N^c + 1) \leq T^c_m\varphi_2 - \varphi_2 \leq (T^c_m\varphi_1 - \varphi_1) + (2N^c + 1).$$

Hence, we have

$$\beta_m(x) - \alpha_m(x) \leq 4N^c + 2, \quad \forall x \in M.$$

For $n \in \mathbb{N}$, $n \geq m$, we have $n = qm + r$, where $0 \leq r < m$. By definition, for any $\varphi \in \text{Lip}(l_1^c)$, we have

$$\alpha_m(x) \leq T_m^c \varphi(x) - \varphi(x) \leq \beta_m(x), \quad \forall x \in M.$$

For $p = 1, 2, \cdots, q$, we have

$$\alpha_m(x) \leq T_{pm}^c \varphi(x) - T_{(p-1)m}^c \varphi(x) \leq \beta_m(x), \quad \forall x \in M.$$

When we sum p from 1 to q, we get

$$q\alpha_m(x) \leq T_{qm}^c \varphi(x) - \varphi(x) \leq q\beta_m(x), \quad \forall x \in M. \tag{3.6}$$

By (3.6), we have

$$q\alpha_m(x) \leq T_{qm+r}^c \varphi(x) - T_r^c \varphi(x) \leq q\beta_m(x), \quad \forall x \in M.$$

Taking $m = 1$ and $q = r$ in (3.6), we get

$$r\alpha_1(x) \leq T_r^c \varphi(x) - \varphi(x) \leq r\beta_1(x), \quad \forall x \in M.$$

Adding the above two inequalities and dividing by $n = qm + r$, we get

$$\frac{q\alpha_m(x) + r\alpha_1(x)}{n} \leq \frac{T_n^c \varphi(x) - \varphi(x)}{n} \leq \frac{q\beta_m(x) + r\beta_1(x)}{n}, \quad \forall x \in M.$$

Note that the difference $\beta_m(x) - \alpha_m(x) \leq 4N^c + 2$, which is independent of m. Let $m \to +\infty$. Then the limit $\lim_{n \to +\infty} T_n^c \varphi(x)/n$ exists. Next, we show that this limit depends only on c.

Fix $\varphi_0 \in \text{Lip}(l_1^c)$. For any $\varphi \in C(M, \mathbb{R})$, there is $n_1, n_2 \in \mathbb{Z}$ such that

$$\varphi_0(x) + n_1 \leq \varphi(x) \leq \varphi_0(x) + n_2, \quad \forall x \in M.$$

Using Proposition 2.9, we have

$$T_t^c(\varphi_0 + n_1)(x) \leq T_t^c \varphi(x) \leq T_t^c(\varphi_0 + n_2)(x), \quad \forall x \in M.$$

By Lemma 3.1, we get

$$\lim_{n \to \infty} \frac{T_n^c \varphi_0(x)}{n} = \lim_{n \to \infty} \frac{T_n^c \varphi(x)}{n}, \quad \forall x \in M.$$

Thus, the limit $\lim_{n \to +\infty} T_n^c \varphi(x)/n$ does not depend on φ.

By Lipschitz continuity, for any $x, y \in M$, we have

$$\lim_{n \to \infty} \frac{T_n^c \varphi_0(x) - l_1^c \text{diam}(M)}{n} \leq \lim_{n \to \infty} \frac{T_n^c \varphi_0(y)}{n} \leq \lim_{n \to \infty} \frac{T_n^c \varphi_0(x) + l_1^c \text{diam}(M)}{n},$$

Thus, the limit $\lim_{n \to +\infty} T_n^c \varphi(x)/n$ does not depend on x.

We denote $t = [t] + \{t\}$, where the integral part $[t] = n$. Note that the limit $\lim_{n \to +\infty} T_n^c \varphi(x)/n$ does not depend on the initial function, we have

$$\lim_{t \to +\infty} \frac{T_t^c \varphi(x)}{t} = \lim_{t \to +\infty} \frac{T_{[t]}^c \circ T_{\{t\}}^c \varphi(x)}{[t]} \frac{[t]}{t} = \lim_{n \to +\infty} \frac{T_n^c \varphi(x)}{n}.$$

We denote by $\rho(c)$ the limit $\lim_{t\to+\infty}\frac{T^c_t\varphi(x)}{t}$, which depends only on c.

(2) Note that for any $\varphi_0\in \mathrm{Lip}(l_1^c)$, we have

$$n\rho(c)=\lim_{m\to+\infty}\frac{T^c_{nm}\varphi_0(z)-\varphi_0(z)}{m}=\lim_{m\to+\infty}\frac{1}{m}\sum_{i=0}^{m-1}(T^c_n-id)(T^c_{in}\varphi_0(z)),\quad \forall z\in M.$$

Then

$$\begin{aligned}|T^c_n\varphi_0(x)-(\varphi_0(x)+n\rho(c))|&=\left|(T^c_n-id)\varphi_0(x)-\lim_{m\to\infty}\frac{1}{m}\sum_{i=0}^{m-1}(T^c_n-id)T^c_{in}\varphi_0(x)\right|\\&\le\beta_n(x)-\alpha_n(x)\le 4N^c+2,\quad\forall n\ge 1.\end{aligned}\tag{3.7}$$

Therefore, $|T^c_t\varphi_0(x)-\rho(c)t|$ is bounded on $M\times[1,+\infty)$ by a constant depending only on c. For any $\varphi\in C(M,\mathbb{R})$, since M is compact, one can show $|T^c_t\varphi(x)-\rho(c)t|$ is bounded on $M\times[1,+\infty)$ by a constant depending only on c by using (3.7).

(3) Now let us consider the properties of $\rho(c)$. From Proposition 2.6, one deduce that the function $c\mapsto\rho(c)$ is nondecreasing. For any $c',c''>c_2$ with $c'>c''$, any $x\in M$, by Proposition 2.7, we get

$$0<T^{c'}_n\varphi(x)-T^{c''}_n\varphi(x)\le ne^{\lambda n}\left(c'-c''\right),\quad\forall\varphi\in\mathrm{Lip}\left(l_1^c\right),\forall x\in M$$

By the definitions of $\alpha_m(x)$ and $\beta_m(x)$, for any given $n\in\mathbb{N}$, we have

$$\alpha_n(x)\le T^{c'}_n\varphi(x)-\varphi(x)\le\beta_n(x)$$

Hence, we have

$$\alpha_n(x)-ne^{\lambda n}\left(c'-c''\right)\le T^{c''}_n\varphi(x)-\varphi(x)\le\beta_n(x)$$

For any given $k\in\mathbb{N}_+$, note that

$$T^c_{kn}\varphi(x)-\varphi(x)=\sum_{j=0}^{k-1}T^c_n\circ T^c_{jn}\varphi(x)-T^c_{jn}\varphi(x)$$

where $T^c_{jn}\varphi(x)\in\mathrm{Lip}\left(l_1^c\right)$ for each j. Then we get

$$k\alpha_n(x)-kne^{\lambda n}\left(c'-c''\right)\le T^{c^*}_{kn}\varphi(x)-\varphi(x)\le k\beta_n(x),\quad\forall\varphi\in\mathrm{Lip}\left(l_1^c\right)$$

which holds true for both $c^*=c'$ or $c^*=c''$. We have proved that the limit $\lim_{n\to+\infty}T^c_t\varphi(x)/t$ exists. Thus, we have

$$\rho(c)=\lim_{k\to+\infty}\frac{T^c_{kn}\varphi(x)}{kn}$$

Then

$$\frac{\alpha_n(x)-ne^{\lambda n}\left(c'-c''\right)}{n}\le\rho\left(c^*\right)\le\frac{\beta_n(x)}{n}$$

Hence, we get

$$0\le\rho\left(c'\right)-\rho\left(c''\right)\le\frac{4N^{c'}+2+ne^{\lambda n}\left(c'-c''\right)}{n}.$$

If c',c'' is contained in a compact interval $I\subset(c_2,+\infty)$, by Corollary 3.1, $N^{c'}$ is bounded by a constant N_I depending only on I. Define $N:=[\frac{c'-c''}{e^{-\lambda}}]$, the solution of $te^{\lambda t}(c'-c''-Ne^{-\lambda})=1$ is no less than 1.

The solution can be expressed as $t = \frac{1}{\lambda}W(\frac{\lambda}{c'-c''-Ne^{-\lambda}})$, where W is the Lambert function. In view of the arbitrariness of n, take $n = [t]$. Thus, we get that

$$\rho(c') - \rho(c'' + Ne^{-\lambda}) \leq \frac{4N^{c'} + 3}{[\frac{1}{\lambda}W(\frac{\lambda}{c'-c''-Ne^{-\lambda}})]}.$$

Note that

$$\begin{aligned}\rho(c') - \rho(c'') =& \rho(c') - \rho(c'' + Ne^{-\lambda}) + \sum_{k=1}^{N}\big(\rho(c'' + ke^{-\lambda}) - \rho(c'' + (k-1)e^{-\lambda})\big) \\ \leq& \frac{4N_I + 3}{[\frac{1}{\lambda}W(\frac{\lambda}{c'-c''-Ne^{-\lambda}})]} + N(4N_I + 3),\end{aligned}$$

The modulus of continuity is defined by

$$\omega(r) := (4N_I + 3)\left[\frac{1}{[\frac{1}{\lambda}W(\frac{\lambda}{r-[\frac{r}{e^{-\lambda}}]e^{-\lambda}})]} + [\frac{r}{e^{-\lambda}}]\right].$$

It is easy to check that $\omega(r)$ is nondecreasing and satisfies $\lim_{r\to 0+}\omega(r) = 0$, which completes the proof.

Acknowledgements

Kaizhi Wang was partly supported by National Natural Science Foundation of China (Grant No. 12171315 and No. 11931016), and Innovation Program of Shanghai Municipal Education Commission No. 2021-01-07-00-02-E00087. Jun Yan was partly supported by National Natural Science Foundation of China (Grant No. 11790272 and No. 11631006).

References

[1] P. Cannarsa, W. Cheng, K. Wang and J. Yan, Herglotz' generalized variational principle and contact type Hamilton-Jacobi equations, Trends in Control Theory and Partial Differential Equations, 39-67. Springer INdAM Ser., 32, Springer, Cham, 2019.

[2] P. Cannarsa, W. Cheng, L. Jin, K. Wang, J. Yan, Herglotz' variational principle and Lax-Oleinik evolution, J. Math. Pures Appl. **141** (2020), 99-136.

[3] M. Crandall and P.-L. Lions, Viscosity solutions of Hanilton-Jacobi equations, Trans. Amer. Math. Soc. **277** (1983), 1-42.

[4] X. Cheng, D. Li, C. Quan and W. Yang. *On a parabolic Sine-Gordan model*. Numerical Mathematics Theory Methods and Applications, **14**(4): 1068-1084, 2021.

[5] M. Bardi and I. Capuzzo-Dolcetta, Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations, Systems & Control: Foundations & Applications. Birkhäuser, 1997.

[6] A. Fathi, *Weak KAM Theorems in Lagrangian Dynamics*. Cambridge: Cambridge University Press, 10th preliminary version, 2008.

[7] A. Fathi, Weak KAM theory: the connection between Aubry-Mather theory and viscosity solutions of the Hamilton-Jacobi equation. Proceedings of the International Congress of MathematiciansłSeoul 2014. Vol. III, 597C621, Kyung Moon Sa, Seoul, 2014.

[8] A. Davini, A. Fathi, R. Iturriaga and M. Zavidovique, Convergence of the solutions of the discounted Hamilton-Jacobi equation: convergence of the discounted solutions, Invent. Math. **206** (2016), 29-55.

[9] G. Herglotz, Berührungstransformationen, Lectures at the University of Göttingen, Gttingen 1930.

[10] G. Herglotz, Gesammelte Schriften. (German) [Collected works] With introductory articles by Peter Bergmann, S. S. Chern, Ronald B. Guenther, Claus Müller, Theodor Schneider and H. Wittich. Edited and with a foreword by Hans Schwerdtfeger. Vandenhoeck & Ruprecht, Göttingen, 1979.

[11] J. Hong, W. Cheng, S. Hu, K. Zhao, Representation formulas for contact type Hamilton-Jacobi equations, arXiv:1907.07542.

[12] H. Ishii, Asymptotic solutions for large time of Hamilton-Jacobi equations, International Congress of Mathematicians. Vol. III, 213-227, Eur. Math. Soc., Zurich, 2006.

[13] P.-L. Lions, G. Papanicolaou and S. Varadhan, Homogenization of Hamilton-Jacobi equation, unpublished preprint, 1987.

[14] P.-L. Lions, Generalized solutions of Hamilton-Jacobi equations, Pitman, Boston, 1982.

[15] R. Mañé, Lagrangain flows: the dynamics of globally minimizing orbits, Bol. Soc. Brasil. Math. **28** (1997), 141-153.

[16] S. Marò and A. Sorrentino, Aubry-Mather theory for conformally symplectic systems, Commun. Math. Phys. **354** (2017), 775-808.

[17] J. Mather, Action minimizing invariant measures for positive definite Lagrangian systems, Math. Z. **207** (1991), 169-207.

[18] H. Mitake and K. Soga, Weak KAM theory for discounted Hamilton-Jacobi equations and its application, Calc. Var. Partial Differential Equations **57** (2018), no. 3, Paper No. 78, 32 pp.

[19] K. Wang and J. Yan, A new kind of Lax-Oleinik type operator with parameters for time-periodic positive definite Lagrangian systems. Comm. Math. Phys. **309** (2012), 663-691.

[20] K. Wang and J. Yan, The rate of convergence of new Lax-Oleinik type operators for time-periodic positive definite Lagrangian systems. Nonlinearity **25** (2012), 2039-2057.

[21] K. Wang, L. Wang and J. Yan, Implicit variational principle for contact Hamiltonian systems, Nonlinearity **30** (2017), 492-515.

[22] K. Wang, The asymptotic bounds of viscosity solutions of the Cauchy problem for Hamilton-Jacobi equations, Pacific J. Math. **298** (2019), 217-232.

[23] K. Wang, L. Wang and J. Yan, Variational principle for contact Hamiltonian systems and its applications, J. Math. Pures Appl. **123** (2019), 167-200.

[24] K. Wang, L. Wang and J. Yan, Aubry-Mather theory for contact Hamiltonian systems, Commun. Math. Phys. **366** (2019), 981-1023.