Multiple asymptotic behaviors of solutions in the generalized vanishing discount problem

Panrui Ni

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Abstract

Consider the generalized discounted Hamilton-Jacobi equation

$$\lambda a(x)u + H(x, Du) = c(H),$$

where $a(x)$ may vanish or change the signs. Two examples are given in this paper showing that the viscosity solutions of the above equation may not converge as λ tends to zero.

Keywords. Hamilton-Jacobi equations; viscosity solutions; vanishing discount problem

1 Introduction and main results

The so-called ergodic approximation is a technique introduced in [10] to study the existence of viscosity solutions of the Hamilton-Jacobi equation

$$H(x, Du(x)) = c \tag{1.1}$$

on the standard torus $\mathbb{T}^n \simeq \mathbb{R}^n/\mathbb{Z}^n$. Here, $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function called the Hamiltonian, and coercive in the second variable, D stands for the space gradient with respect to x, and c is a real number. Let $\lambda > 0$ and u_λ be the unique viscosity solution of

$$\lambda u(x) + H(x, Du(x)) = 0.$$

According to [10], there is a sequence $\lambda_k \to 0+$ such that $-\lambda_k u_{\lambda_k}(x)$ uniformly converges to a constant $c(H)$ and $u_{\lambda_k} - \min_{x\in M} u_{\lambda_k}(x)$ uniformly converges to a viscosity solution of (1.1) with $c = c(H)$. Moreover, $c(H)$ is the unique value such that (1.1) admits viscosity solutions.

Shanghai Center for Mathematical Sciences, Fudan University, Shanghai 200433, China. e-mail: prni18@fudan.edu.cn

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One can also refer to [6, Section 3] for a short self-contained proof. The constant $c(H)$ is called the Mañé critical value of H, which is characterized by

$$c(H) = \min\{c \in \mathbb{R} : \ (1.1) \text{ admits subsolutions}\}.$$

Later, [4] proved the uniform convergence of the unique viscosity solution u_λ of

$$\lambda u(x) + H(x, Du(x)) = c(H), \quad x \in M$$

as $\lambda \to 0+$, where M is a compact connected manifold without boundary, and H is a continuous Hamiltonian defined on the cotangent bundle over M, coercive and convex in the fibre. When the convexity of H does not hold, [18] gives a counterexample showing that u_λ does not converge. In [5], the authors discussed the uniform convergence of the minimal viscosity solution of the equation above as $\lambda \to 0-$. As a nonlinear generalization, [1, 2] and [16] consider the uniform convergence of the unique viscosity solution of

$$H_\lambda(x, Du(x), u(x)) = c(G), \quad x \in M \tag{1.2}$$

as $\lambda \to 0+$, where $H_\lambda(x, p, u)$ is strictly increasing in u, and uniformly converges to $G(x, p)$ as $\lambda \to 0+$. As a degenerate case, [17] considers the convergence of the viscosity solution of

$$\lambda a(x) u(x) + H(x, Du(x)) = c(H), \quad x \in M \tag{1.3}$$

as $\lambda \to 0+$, where $a(x) \geq 0$ on M, and $a(x) > 0$ on the projected Aubry set of H.

The present paper provides two examples showing that the viscosity solutions of the generalized discounted equation (1.3) may not converge. The first example comes from the non-monotone model discussed in [9] and [12]. Theorem 1 shows that the asymptotic behavior of the viscosity solutions of (1.3) can be complicated. So one can not expect that the results in the classical vanishing discount problem still hold in the non-monotone cases.

Let $x \in \mathbb{S}^1 \simeq [0, 2\pi)$. Define

$$H(x, p) = p^2 + \sin x - 1. \tag{1.4}$$

By [5, Proposition 7], we have $c(H) = 0$ and the projected Aubry set of H is $\{\pi/2\}$.

Theorem 1. *In (1.3), we take $a(x) = \sin x$, and $H(x, p)$ is given by (1.4). For $\lambda \in \mathbb{R}\backslash\{0\}$, (1.3) becomes*

$$\lambda \sin x \cdot u(x) + (u'(x))^2 + \sin x - 1 = 0. \tag{1.5}$$

Then

(1) let $\mathcal{S}_\lambda$ be the set of the viscosity solutions of (1.5). When $|\lambda| < W(\frac{2}{\pi^2+2})$, we have $\mathcal{S}_\lambda = \{u_\lambda, v_\lambda\}$, where W is the inverse function of xe^x.

(2) the family $\{u_\lambda\}$ with $u_\lambda(\frac{\pi}{2}) = 0$ uniformly converges to a viscosity solution of

$$(u'(x))^2 + \sin x - 1 = 0 \tag{1.6}$$

as λ tends to zero;

(3) *the family* $\{v_\lambda\}$ *with* $v_\lambda(\frac{3\pi}{2}) = -\frac{2}{\lambda}$ *uniformly converges to* $-\infty$ *(resp.* $+\infty$*) as* $\lambda \to 0+$ *(resp.* $\lambda \to 0-$*). Moreover, the family* $\{v_\lambda + \frac{2}{\lambda}\}$ *uniformly converges to a viscosity solution of*

$$(u'(x))^2 - \sin x - 1 = 0 \tag{1.7}$$

as λ *tends to zero.*

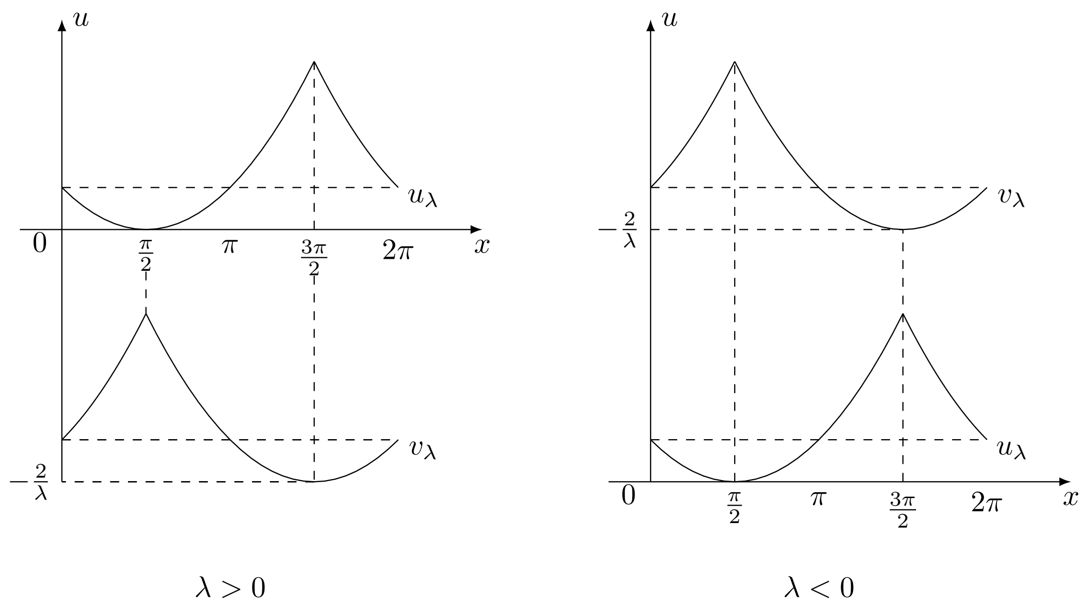

Figure 1: The solutions of (1.5) when $|\lambda| < W(\frac{2}{\pi^2+2})$.

Remark 1.1. *Since Proposition 2.1 below is established under the assumption that* M *is an* n*-dimensional manifold, we can generalize the conclusions in Theorem 1 (2) and (3). Consider*

$$\lambda \sin(x_1 + \cdots + x_n) \cdot u(x) + \sum_{i=1}^{n} (\partial_{x_i} u(x))^2 + \sin(x_1 + \cdots + x_n) - 1 = 0, \tag{1.8}$$

where $x = (x_1, \ldots, x_n) \in \mathbb{T}^n \simeq [0, 2\pi)^n$*. Then* $\varphi_0 \equiv 0$ *and* $\varphi_\lambda \equiv -\frac{2}{\lambda}$ *are two subsolutions of (1.8). Similar to the proof at the end of Section 2.1, the family*

$$\{u_\lambda := \lim_{t \to +\infty} T_t^\lambda \varphi_0\}$$

uniformly converges to a viscosity solution of

$$\sum_{i=1}^{n} (\partial_{x_i} u(x))^2 + \sin(x_1 + \cdots + x_n) - 1 = 0,$$

while the family

$$\{v_\lambda := \lim_{t \to +\infty} T_t^\lambda \varphi_\lambda\}$$

is divergent as λ *tends to zero. Here both* u_λ *and* v_λ *are viscosity solutions of (1.8).*

The second example shows that the convergence of the viscosity solutions of (1.3) does not hold when $a(x) \geq 0$ and $a(x) = 0$ for some x in the projected Aubry set of H.

Proposition 1.1. *Let $a(x) = 1 - \sin x$ and $H(x,p)$ be given by (1.4). For $\lambda > 0$, (1.3) becomes*

$$\lambda(1 - \sin x) \cdot u(x) + (u'(x))^2 + \sin x - 1 = 0. \tag{1.9}$$

The function $a(x)$ satisfies $a(x) \geq 0$ and $a(\pi/2) = 0$. One can check that the constant function $v_\lambda \equiv 1/\lambda$ is a classical solution of (1.9), which tends to $+\infty$ as $\lambda \to 0+$. At the same time, the constant function $\varphi_0 \equiv 0$ is a subsolution of (1.9). By Proposition 2.1 below, there is a family

$$\{u_\lambda := \lim_{t \to +\infty} T_t^\lambda \varphi_0\}$$

uniformly converges to a viscosity solution of (1.6), where u_λ is a viscosity solution of (1.9).

2 Proof of Theorem 1

2.1 Existence of the convergent family

In this section, we prove the existence of the convergent families $\{u_\lambda\}$ and $\{v_\lambda + \frac{2}{\lambda}\}$ in Theorem 1. In order to show the generality of this phenomenon, we will prove this result under more general assumptions.

Assume that M is a connected, closed (compact without boundary) and smooth Riemannian manifold. Denote by TM and T^*M the tangent and cotangent bundle over M respectively. Let $H : T^*M \times \mathbb{R} \to \mathbb{R}$ be a C^3 function satisfying

(H1) *Strict convexity*: $\frac{\partial^2 H}{\partial p^2}(x,p,u)$ is positive definite for all $(x,p,u) \in T^*M \times \mathbb{R}$;

(H2) *Superlinearity*: for every $(x,u) \in M \times \mathbb{R}$, $H(x,p,u)$ is superlinear in p;

(H3) *Lipschitz continuity*: $|\frac{\partial H}{\partial u}(x,p,u)| \leq 1$ for all $(x,p,u) \in T^*M \times \mathbb{R}$.

where (H1) and (H2) are referred to as the *Tonelli conditions*. The Lagrangian associated with $H(x,p,u)$ is defined as

$$L(x,\dot{x},u) := \sup_{p \in T_x^*M} \{\langle \dot{x}, p \rangle_x - H(x,p,u)\},$$

where $\langle \cdot, \cdot \rangle_x$ represents the canonical pairing between the tangent space and cotangent space. According to [13, Page 494], the Lagrangian $L : TM \times \mathbb{R} \to \mathbb{R}$ satisfies

(L1) *Strict convexity*: $\frac{\partial^2 L}{\partial \dot{x}^2}(x,\dot{x},u)$ is positive definite for all $(x,\dot{x},u) \in TM \times \mathbb{R}$;

(L2) *Superlinearity*: for every $(x,u) \in M \times \mathbb{R}$, $L(x,\dot{x},u)$ is superlinear in $\dot{x}$;

(L3) *Lipschitz continuity*: $|\frac{\partial L}{\partial u}(x,\dot{x},u)| \leq 1$ for all $(x,\dot{x},u) \in TM \times \mathbb{R}$.

We denote by $C(M,\mathbb{R})$ the space of real valued continuous functions on M. According to [14], there is a unique semigroup of operators $\{T_t\}_{t\geq 0}: C(M,\mathbb{R})^{\circlearrowleft}$ such that

$$T_t\varphi(x) = \inf_{\gamma(t)=x}\left\{\varphi(\gamma(0)) + \int_0^t L(\gamma(\tau),\dot\gamma(\tau),T_\tau\varphi(\gamma(\tau)))\mathrm{d}\tau\right\}, \tag{2.1}$$

where the infimum is taken among the Lipschitz continuous curves $\gamma:[0,t]\to M$ with $\gamma(t)=x$ and can be achieved. For each $\varphi\in C(M,\mathbb{R})$, the function $(x,t)\mapsto T_t\varphi(x)$ is the unique viscosity solution of

$$\begin{cases} \partial_t u(x,t) + H(x,Du(x,t),u(x,t)) = 0, \quad (x,t)\in M\times(0,+\infty). \\ u(x,0)=\varphi(x), \quad x\in M. \end{cases}$$

In [15], the authors defined the weak KAM solutions of

$$H(x,Du(x),u(x)) = 0, \quad x\in M. \tag{2.2}$$

Definition 2.1. *A function $u\in C(M,\mathbb{R})$ is called a backward weak KAM solution of (2.2) if*

(1) For each continuous piecewise C^1 curve $\gamma:[t',t]\to M$, we have

$$u(\gamma(t)) - u(\gamma(t')) \leq \int_{t'}^t L(\gamma(s),\dot\gamma(s),u(\gamma(s)))ds.$$

The above condition reads that u is dominated by L and denoted by $u\prec L$.

(2) For each $x\in M$, there exists a C^1 curve $\gamma_-:(-\infty,0]\to M$ with $\gamma_-(0)=x$ such that

$$u(x) - u(\gamma_-(t)) = \int_t^0 L(\gamma_-(s),\dot\gamma_-(s),u(\gamma_-(s)))ds, \quad \forall t<0.$$

The curves satisfying the above equality are called $(u,L,0)$-calibrated curves.

According to [15, Proposition 2.7] (see also [11, Proposition D.4]), the following statements are equivalent: u is a backward weak KAM solution of (2.2); u is a fixed point of T_t; u is a viscosity solution of (2.2).

Let $\gamma:(-\infty,0]\to M$ be a $(u,L,0)$-calibrated curve. According to [15, Lemma 4.3], u is differentiable on $\gamma(t)$ for $t\in(-\infty,0)$, and $Du(\gamma(t)) = \frac{\partial L}{\partial\dot x}(\gamma(t),\dot\gamma(t),u(\gamma(t)))$. Moreover, since calibrated curves are minimizers, the orbit $(\gamma(t),Du(\gamma(t)),u(\gamma(t)))$ satisfies the contact Hamilton equations

$$\begin{cases} \dot x = \frac{\partial H}{\partial p}(x,p,u), \\ \dot p = -\frac{\partial H}{\partial x}(x,p,u) - \frac{\partial H}{\partial u}(x,p,u)p, \\ \dot u = \frac{\partial H}{\partial p}(x,p,u)\cdot p - H(x,p,u). \end{cases} \tag{2.3}$$

for $t\in(-\infty,0)$, see [13, Theorem A].

Proposition 2.1. *Assume that there is $k \in \mathbb{R}$ such that $H(x,0,k) \leq c_k$, where c_k is the Mañé critical value of $H(x,p,k)$. Then for each $\lambda \in \mathbb{R}\backslash\{0\}$, let T_t^λ be the semigroup defined in (2.1) associated with the Lagrangian $L(x,\dot{x},\lambda u)+c_k$, the uniform limit*

$$w_\lambda := \lim_{t\to+\infty} T_t^\lambda\left(\frac{k}{\lambda}\right)$$

exists, and is a viscosity solution of

$$H(x,Du(x),\lambda u(x)) = c_k, \quad x \in M. \tag{2.4}$$

The family $\{w_\lambda - \frac{k}{\lambda}\}$ uniformly converges to a viscosity solution w_0 of

$$H(x,Du,k) = c_k. \tag{2.5}$$

Moreover, the solution w_0 is the unique solution which equals to zero on the projected Aubry set associated to (2.5).

Definition 2.2. *The projected Aubry set $\mathcal{A} \subset M$ associated to (2.5) can be characterized by the following property, see [8]*

$$x \in \mathcal{A} \quad \textit{iff} \quad \textit{any subsolutions of (2.5) is differentiable at } x.$$

The projected Aubry set is very important in the Aubry-Mather theory and weak KAM theory, see [6, Section 5] and [7, Section 8]. For the properties of $\mathcal{A}$, one can refer to [5, Propostion 5].

If $H(x,0,k) \leq c_k$, it is direct to see that the constant function $\varphi_k := \frac{k}{\lambda}$ is a viscosity subsolution of (2.4). According to [12, Proposition 2.5], $T_t^\lambda\varphi_k$ is nondecreasing in t. If $T_t^\lambda\varphi_k$ is bounded from above, then by [14, Proof of Theorem 1.2], the limit procedure $\lim_{t\to+\infty} T_t^\lambda\varphi_k$ is uniform, and w_λ is a viscosity solution of (2.4).

In Lemmas 2.1 and 2.2, we prove that $T_t^\lambda\varphi_k$ is bounded from above. In Lemma 2.3, we show that $\{w_\lambda - \frac{k}{\lambda}\}_{|\lambda|\leq 1}$ is uniformly bounded and equi-Lipschitz continuous. In Lemma 2.4, we prove that $\{w_\lambda - \frac{k}{\lambda}\}$ uniformly converges to a solution of (2.5).

Lemma 2.1. *Let $\mathcal{A}$ be the projected Aubry set associated with (2.5). For all $y \in \mathcal{A}$, we have $T_t^\lambda\varphi_k(y) = \frac{k}{\lambda}$ for all $t \geq 0$.*

Proof. By [5, Proposition 5], for $y \in \mathcal{A}$, there is a unique curve $\gamma : \mathbb{R} \to \mathcal{A} \subset M$ with $\gamma(0) = y$ such that

$$v(\gamma(b)) - v(\gamma(a)) = \int_a^b \big[L(\gamma(s),\dot{\gamma}(s),k) + c_k\big]ds, \quad \forall a < b,$$

for any subsolution v of (2.5). Since v is differentiable on $\mathcal{A}$, γ is of class C^1, and $H(x,Dv,k) = c_k$ on $\mathcal{A}$, we have

$$Dv(\gamma(s)) = \frac{\partial L}{\partial \dot{x}}(\gamma(s),\dot{\gamma}(s),k), \quad \forall s.$$

Since $H(x,0,k) \leq c_k$, and $Dv(y) = Dw(y)$ for each $y \in \mathcal{A}$ and any pair v,w of subsolutions of (2.5), we have

$$\frac{\partial L}{\partial \dot{x}}(\gamma(s),\dot{\gamma}(s),k) = 0, \quad \forall s.$$

Since the energy of γ equals to c_k (cf. [3, 7]), we have

$$c_k=\left\langle \dot\gamma(s),\frac{\partial L}{\partial \dot x}(\gamma(s),\dot\gamma(s),k)\right\rangle_{\gamma(s)}-L(\gamma(s),\dot\gamma(s),k)=-L(\gamma(s),\dot\gamma(s),k).$$

Recall that $T^\lambda_t\varphi_k\geq\frac{k}{\lambda}$ for all $t\geq 0$. By contradiction, we assume that there is $t>0$ such that $T^\lambda_t\varphi_k(y)>\frac{k}{\lambda}$. Taking the curve $\beta(\tau)=\gamma(\tau-t)$ for $\tau\in[0,t]$. By continuity, there is $\sigma\in[0,t)$ such that $T^\lambda_\sigma\varphi_k(\beta(\sigma))=\frac{k}{\lambda}$ and $T^\lambda_s\varphi_k(\beta(s))>\frac{k}{\lambda}$ for all $s\in(\sigma,t]$. By the definition of the semigroup, we have

$$\begin{aligned}T^\lambda_s\varphi_k(\beta(s))&\leq T^\lambda_\sigma\varphi_k(\beta(\sigma))+\int_\sigma^s\big[L(\beta(\tau),\dot\beta(\tau),\lambda T^\lambda_\tau\varphi_k(\beta(\tau)))+c_k\big]d\tau\\&\leq\frac{k}{\lambda}+\int_\sigma^s\big[L(\beta(\tau),\dot\beta(\tau),k)+c_k+|\lambda|(T^\lambda_\tau\varphi_k(\beta(\tau))-\frac{k}{\lambda})\big]d\tau\\&=\frac{k}{\lambda}+|\lambda|\int_\sigma^s(T^\lambda_\tau\varphi_k(\beta(\tau))-\frac{k}{\lambda})d\tau.\end{aligned}$$

By the Gronwall inequality, we have $T^\lambda_s\varphi_k(\beta(s))-\frac{k}{\lambda}=0$ for all $s\in[\sigma,t]$, which contradicts $T^\lambda_t\varphi_k(y)>\frac{k}{\lambda}$. □

In the following, we denote by $\|\cdot\|$ the norms induced by the Riemannian metric g on both tangent and cotangent spaces of M, and diam(M) the diameter of M.

Lemma 2.2. *For all $t\geq 1$, we have $T^\lambda_t\varphi_k\leq\frac{k}{\lambda}+C_Le^{|\lambda|}$, where*

$$C_L:=\sup_{x\in M,\|\dot x\|\leq diam(M)}|L(x,\dot x,k)+c_k|.$$

Proof. We take an arbitrary point $x\in M$. Let $\alpha:[0,1]\to M$ be a geodesic satisfying $\alpha(0)=y\in\mathcal{A}$ and $\alpha(1)=x$ with constant speed. Then $\|\dot\alpha\|\leq$ diam(M). Let $t\geq 1$. If $T^\lambda_t\varphi_k(x)=\frac{k}{\lambda}$, then the proof is finished. If $T^\lambda_t\varphi_k(x)>\frac{k}{\lambda}$, since $T^\lambda_{t-1}\varphi_k(y)=\frac{k}{\lambda}$ by Lemma 2.1, there is $\sigma\in[0,1)$ such that $T^\lambda_{t-1+\sigma}\varphi_k(\alpha(\sigma))=\frac{k}{\lambda}$ and $T^\lambda_{t-1+s}\varphi_k(\alpha(s))>\frac{k}{\lambda}$ for all $s\in(\sigma,1]$. By the definition of the semigroup, we have

$$\begin{aligned}T^\lambda_{t-1+s}\varphi_k(\alpha(s))&\leq T^\lambda_{t-1+\sigma}\varphi_k(\alpha(\sigma))+\int_\sigma^s\big[L(\alpha(\tau),\dot\alpha(\tau),\lambda T^\lambda_{t-1+\tau}\varphi_k(\alpha(\tau)))+c_k\big]d\tau,\\&\leq\frac{k}{\lambda}+C_L(s-\sigma)+|\lambda|\int_\sigma^s(T^\lambda_{t-1+\tau}\varphi_k(\alpha(\tau))-\frac{k}{\lambda})d\tau.\end{aligned}$$

By the Gronwall inequality we get

$$T^\lambda_{t-1+s}\varphi_k(\alpha(s))-\frac{k}{\lambda}\leq C_Le^{|\lambda|(s-\sigma)}\leq C_Le^{|\lambda|},\quad\forall s\in(\sigma,1].$$

Take $s=1$, we have $T^\lambda_t\varphi_k(x)\leq\frac{k}{\lambda}+C_Le^{|\lambda|}$. □

Lemma 2.3. *The family $\{w_\lambda-\frac{k}{\lambda}\}_{|\lambda|\leq 1}$ is uniformly bounded and equi-Lipschitz continuous.*

Proof. By Lemma 2.2, we have

$$0 \leq w_\lambda - \frac{k}{\lambda} \leq C_L e^{|\lambda|} \leq C_L e,$$

which implies that $\{w_\lambda - \frac{k}{\lambda}\}_{|\lambda|\leq 1}$ is uniformly bounded.

For each $x, y \in M$, we denote by $d := d(x,y)$ the distance between them. Take a geodesic $\alpha : [0,d] \to M$ satisfying $\alpha(0) = x$ and $\alpha(d) = y$ with constant speed $\|\dot{\alpha}\| = 1$. Since $w_\lambda \prec L(x, \dot{x}, \lambda u) + c_k$, we have

$$\begin{aligned} w_\lambda(y) - w_\lambda(x) &\leq \int_0^d \big[L(\alpha(\tau), \dot{\alpha}(\tau), \lambda w_\lambda(\alpha(\tau))) + c_k\big] d\tau \\ &\leq \int_0^d \big[L(\alpha(\tau), \dot{\alpha}(\tau), k) + c_k + |\lambda|(w_\lambda - \frac{k}{\lambda})\big] d\tau \leq (K_L + C_L e) d(x,y), \end{aligned}$$

where

$$K_L := \sup_{x\in M, \|\dot{x}\|\leq 1} |L(x, \dot{x}, k) + c_k|.$$

Exchanging the role of x and y, the proof is complete. □

Lemma 2.4. *The family $\{w_\lambda - \frac{k}{\lambda}\}$ uniformly converges to the unique viscosity solution of (2.5) which equals to zero on $\mathcal{A}$ as $\lambda \to 0$.*

Proof. Since the family $\{w_\lambda - \frac{k}{\lambda}\}_{|\lambda|\leq 1}$ is uniformly bounded and equi-Lipschitz continuous, there is a sequence λ_j converges to zero such that $\{w_{\lambda_j} - \frac{k}{\lambda_j}\}$ uniformly converges to a continuous function u_*. Since w_λ solves (2.4), the function $w_\lambda - \frac{k}{\lambda}$ is a viscosity solution of

$$H(x, Dw, \lambda w + k) = c_k.$$

By the stability of viscosity solutions, u_* is a viscosity solution of (2.5). Let $\mathcal{S}_*$ be the set of such functions u_*, it is sufficient to show that $\mathcal{S}_*$ is a singleton. By Lemma 2.1, $w_\lambda - \frac{k}{\lambda}$ equals to zero on $\mathcal{A}$. Thus, u_* equals to zero on $\mathcal{A}$. By [6, Theorem 6.7], the viscosity solution of (2.5) which equals to zero on $\mathcal{A}$ is unique. □

Now we return to the proof of Theorem 1. Let

$$H(x,p,u) = \sin x \cdot u + p^2 + \sin x - 1,$$

then (1.5) is equivalent to $H(x, u', \lambda u) = 0$.

Since $H(x,0,0) \leq 0$ and the critical value of $H(x,p,0)$ is zero, by Proposition 2.1, the function

$$u_\lambda := \lim_{t\to+\infty} T_t^\lambda \varphi_0$$

is a viscosity solution of (1.5), where $\varphi_0 \equiv 0$. The family $\{u_\lambda\}$ uniformly converges to the unique viscosity solution of (1.6) which equals to zero at $\pi/2$.

Since $H(x,0,-2)\leq 0$ and the critical value of $H(x,p,-2)$ is zero, by Proposition 2.1, the function

$$v_\lambda := \lim_{t\to+\infty} T_t^\lambda\left(-\frac{2}{\lambda}\right)$$

is a viscosity solution of (1.5). The family $\{v_\lambda+\frac{2}{\lambda}\}$ uniformly converges to the unique viscosity solution of (1.7) which equals to zero at $3\pi/2$.

Remark 2.1. *Consider the case discussed in [5], i.e.,*

$$-\lambda u(x)+H(x,Du(x))=c(H),\quad \lambda>0. \tag{2.6}$$

The authors proved that when $H(x,0)\leq c(H)$, the minimal viscosity solution of (2.6) converges. Let $H(x,p,u):=-u+H(x,p)$, then $H(x,0,0)=H(x,0)\leq c(H)$ by the assumption. Let $\varphi_0\equiv 0$. In general, the solution $w_\lambda=\lim_{t\to+\infty}T_t^\lambda\varphi_0$ we get in Proposition 2.1 is different from the minimal viscosity solution. Here is an example. Let $r(x)$ be a smooth function defined on the unit circle $\mathbb{S}^1$, which satisfies $r(x)\leq 0$, $\max_{x\in\mathbb{S}^1} r(x)=0$ and there is a point y such that $r(y)<0$. Let $U(x):=r(x)-(r'(x))^2$ and $H(x,p):=p^2+U(x)$, then $c(H)=0$. Consider the following equation

$$-u(x)+(u'(x))^2+U(x)=0, \tag{2.7}$$

Then $r(x)$ itself is a classical solution of (2.7). Moreover, we have $r(y)<0\leq w_\lambda(y)$, which implies that w_λ is not minimal.

2.2 The structure of $\mathcal{S}_\lambda$

In the following, we define the contact Hamiltonian

$$H_\lambda(x,p,u)=\lambda\sin x\cdot u+p^2+\sin x-1.$$

The corresponding Lagrangian is

$$L_\lambda(x,\dot{x},u)=-\lambda\sin x\cdot u+\frac{\dot{x}^2}{4}-\sin x+1.$$

For each solution w_λ of (1.5), let $\gamma:(-\infty,0]\to\mathbb{S}^1$ be a $(w_\lambda,L_\lambda,0)$-calibrated curve. Similar to the analysis at the beginning of [9, Section 3.2], the derivative $w_\lambda'(\gamma(t))$ exists for each $t\in(-\infty,0)$ and the orbit $(\gamma(t),w_\lambda'(\gamma(t)),w_\lambda(\gamma(t)))$ satisfies the contact Hamilton equations generated by H_λ. Then the analysis of the structure of $\mathcal{S}_\lambda$ is related to the contact Hamiltonian flow Φ_t generated by H_λ.

Since $w_\lambda\in\mathcal{S}_\lambda$, we have

$$H_\lambda(\gamma(t),w_\lambda'(\gamma(t)),w_\lambda(\gamma(t)))=0$$

for $t\in(-\infty,0)$. We then discuss the flow on the two dimensional energy shell

$$M^0:=\{(x,p,u)\in T^*\mathbb{S}^1\times\mathbb{R}:\ H_\lambda(x,p,u)=0\}.$$

Note that along the contact Hamiltonian flow, we have $\frac{dH_\lambda}{dt} = -\frac{\partial H_\lambda}{\partial u} H_\lambda$, which equals to zero on the set M^0. Thus, M^0 is an invariant set under the action of Φ_t. Since we are interested in the orbit $(\gamma(t), w'_\lambda(\gamma(t)), w_\lambda(\gamma(t)))$, we then consider the flow Φ_t restrict on M^0. The contact Hamilton equations (2.3) then reduce to

$$\begin{cases} \dot{x} = 2p, \\ \dot{p} = -(\lambda \cos x \cdot u + \cos x) - \lambda \sin x \cdot p, \\ \dot{u} = 2p^2. \end{cases} \tag{2.8}$$

Lemma 2.5. *The non-wandering set of $\Phi_t|_{M^0}$, which is denoted by Ω, contains only two points*

$$P_1 = (\frac{\pi}{2}, 0, 0), \quad P_2 = (\frac{3\pi}{2}, 0, -\frac{2}{\lambda}).$$

Moreover, P_1 and P_2 are hyperbolic fixed points for the dynamical system $\Phi_t|_{M^0}$.

Proof. Suppose there is an orbit $(x(t), p(t), u(t))$ belongs to Ω. Since $\dot{u} = 2p^2 \geq 0$, $u(t)$ equals to a constant and $p(t) \equiv 0$. By $\dot{x}(t) = 2p(t) = 0$, $x(t)$ also equals to a constant. By $H_\lambda(x, p, u) = 0$ and $p = 0$, we have

$$\lambda \sin x \cdot u + \sin x - 1 = 0.$$

By $p = 0$ and $\dot{p} = 0$ we have

$$\lambda \cos x \cdot u + \cos x = 0.$$

A direct calculation shows that the non-wandering points are P_1 and P_2.

Near the points P_1 and P_2, let $y = x - \pi/2$ (resp. $y = x - 3\pi/2$), the linearised equations of (2.8) are

$$\dot{y} = 2p, \quad \dot{p} = y - \lambda p, \quad \dot{u} = 0$$

and

$$\dot{y} = 2p, \quad \dot{p} = y + \lambda p, \quad \dot{u} = 0$$

respectively. A direct calculation shows that P_1 and P_2 are hyperbolic. □

Let $\gamma : (-\infty, 0] \to \mathbb{S}^1$ be a $(w_\lambda, L_\lambda, 0)$-calibrated curve. Denote by $|\cdot|$ the distance on $\mathbb{S}^1$. We define the α-limit set of γ by

$$\alpha(\gamma) := \{x \in \mathbb{S}^1 : \text{ there exists a sequence } t_n \to -\infty \text{ such that } |\gamma(t_n) - x| \to 0\},$$

Elementary knowledge of topological dynamics shows that for all sequence $t_n \to -\infty$, the limit points of the orbit $(\gamma(t_n), w'_\lambda(\gamma(t_n)), w_\lambda(\gamma(t_n)))$ are contained in Ω. Then the α-limit set of γ is contained in the projection of Ω. Thus, the α-limit set of γ can only be either $\pi/2$ or $3\pi/2$.

Lemma 2.6. *Let w_λ be a viscosity solution of (1.5) and $k \in \mathbb{R}$. If $w_\lambda(x_\lambda) \leq k/\lambda$ for some $x_\lambda \in \mathbb{S}^1$, then*

$$w_\lambda \leq \frac{k}{\lambda} + (\pi^2 + |k+1| + 1)e^{|\lambda|}. \tag{2.9}$$

Proof. Let $x \in \mathbb{S}^1$. If $w_\lambda(x) \le k/\lambda$, then the proof is finished. So we assume $w_\lambda(x) > k/\lambda$. Take a geodesic $\alpha : [0,1] \to \mathbb{S}^1$ with constant speed, and $\alpha(0) = x_\lambda$, $\alpha(1) = x$. Then $|\dot\alpha(s)| \le 2\pi$. By continuity, there is a constant $\sigma \in [0,1)$ such that $w_\lambda(\alpha(\sigma)) = k/\lambda$, and $w_\lambda(\alpha(s)) > k/\lambda$ for all $s \in (\sigma, 1]$. Since $u_\lambda \prec L_\lambda$, we have

$$\begin{aligned} w_\lambda(\alpha(s)) - w_\lambda(\alpha(\sigma)) &\le \int_\sigma^s \left[\frac{\dot\alpha(\tau)^2}{4} - \sin(\alpha(\tau)) + 1 - \lambda \sin(\alpha(\tau)) w_\lambda(\alpha(\tau))\right] d\tau \\ &\le \int_\sigma^s \left[\pi^2 - (k+1)\sin(\alpha(\tau)) + 1 - \lambda \sin(\alpha(\tau))\big(w_\lambda(\alpha(\tau)) - \frac{k}{\lambda}\big)\right] d\tau, \\ &\le \pi^2 + |k+1| + 1 + |\lambda| \int_\sigma^s \left[w_\lambda(\alpha(\tau)) - \frac{k}{\lambda}\right] d\tau, \quad s \in (\sigma, 1]. \end{aligned}$$

By the Gronwall inequality, we have

$$w_\lambda(\alpha(s)) - \frac{k}{\lambda} \le (\pi^2 + |k+1| + 1)e^{|\lambda|}.$$

Take $s = 1$, we obtain (2.9). $\square$

Lemma 2.7. *Let $|\lambda| < W(\frac{2}{\pi^2+2})$. Let $w_\lambda \in \mathcal{S}_\lambda$ and γ_x be a $(w_\lambda, L_\lambda, 0)$-calibrated curve with $\gamma_x(0) = x$. Then one of the following cases holds:*

(1) for all $x \in \mathbb{S}^1$, $\alpha(\gamma_x) = \pi/2$;

(2) for all $x \in \mathbb{S}^1$, $\alpha(\gamma_x) = 3\pi/2$.

Proof. We argue by contradiction. Assume that there are two points $x_1, x_2 \in \mathbb{S}^1$ such that $\alpha(\gamma_{x_1}) = \pi/2$ and $\alpha(\gamma_{x_2}) = 3\pi/2$. There are two cases.

Case 1: $\lambda > 0$. By the last equality of (2.8), w_λ is nondecreasing along γ_{x_1}. Thus, $w_\lambda(x_1) \ge 0$. Note that

$$w_\lambda(\frac{3\pi}{2}) = \lim_{t \to -\infty} w_\lambda(\gamma_{x_2}(t)) = -\frac{2}{\lambda}.$$

Take $k = -2$ in (2.9), we have $w_\lambda < 0$ when $\lambda < W(\frac{2}{\pi^2+2})$, which contradicts $w_\lambda(x_1) \ge 0$.

Case 2: $\lambda < 0$. By the last equality of (2.8), w_λ is nondecreasing along γ_{x_2}. Thus, $w_\lambda(x_2) \ge -2/\lambda$. Note that

$$w_\lambda(\frac{\pi}{2}) = \lim_{t \to -\infty} w_\lambda(\gamma_{x_1}(t)) = 0.$$

Take $k = 0$ in (2.9), we have $w_\lambda \le (\pi^2 + 2)e^{-\lambda}$, which contradicts $w_\lambda(x_2) \ge -2/\lambda$ when $\lambda > -W(\frac{2}{\pi^2+2})$. $\square$

Now we are going to show that $w_\lambda \in \mathcal{S}_\lambda$ with $w_\lambda(\pi/2) = 0$ is unique. Let w_λ satisfy the case (1) in Lemma 2.7. Then $w_\lambda(\pi/2) = 0$.

We first show that w_λ is unique near $\pi/2$. By the discussion above, we know that P_1 is hyperbolic, the orbit $(\gamma_x(t), w'_\lambda(\gamma_x(t)), w_\lambda(\gamma_x(t)))$ satisfies $\Phi_t|_{M^0}$ for each $t \in (-\infty, 0)$, and converges to P_1 as $t \to \infty$. Then the 1-graph $(x, w'_\lambda(x), w_\lambda(x))$ coincides with the local unstable manifold of P_1. Therefore, the solution w_λ is unique on $[\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta]$ for $\delta > 0$ small enough.

Assume that there are two solutions u and v of (1.5) satisfying $u(\pi/2) = v(\pi/2) = 0$ and $u(x) > v(x)$ at some point $x \in \mathbb{S}^1$. Let γ be a $(v, L_\lambda, 0)$-calibrated curve with $\gamma(0) = x$. Then the α-limit point of γ is $\pi/2$. Define $t_0 < 0$ such that $\gamma(t_0) \in [\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta]$, and let

$$G(s) := u(\gamma(s)) - v(\gamma(s)), \quad s \in [t_0, 0].$$

Then $G(t_0) = 0$ and $G(0) > 0$. By continuity, there is $\sigma \in [t_0, 0)$ such that $G(\sigma) = 0$ and $G(s) > 0$ for all $s \in (\sigma, 0]$. By definition we have

$$u(\gamma(s)) - u(\gamma(\sigma)) \leq \int_\sigma^s \left[\frac{\dot{\gamma}(\tau)^2}{4} - \sin(\gamma(\tau)) + 1 - \lambda \sin(\gamma(\tau))u(\gamma(\tau))\right] d\tau,$$

and

$$v(\gamma(s)) - v(\gamma(\sigma)) = \int_\sigma^s \left[\frac{\dot{\gamma}(\tau)^2}{4} - \sin(\gamma(\tau)) + 1 - \lambda \sin(\gamma(\tau))v(\gamma(\tau))\right] d\tau,$$

which implies

$$G(s) \leq |\lambda| \int_\sigma^s G(\tau) d\tau.$$

By the Gronwall inequality, we have $G(s) \equiv 0$ for all $s \in (\sigma, 0]$, which contradicts $u(x) > v(x)$. We conclude that the solution w_λ with $w_\lambda(\pi/2) = 0$ is unique.

By a similar argument, $w_\lambda \in \mathcal{S}_\lambda$ with $w_\lambda(3\pi/2) = -2/\lambda$ is also unique. It is direct to see that the solution w_λ satisfying $w_\lambda(\pi/2) = 0$ (resp. $w_\lambda(3\pi/2) = -2/\lambda$) coincides with the solution u_λ (resp. v_λ) obtained in Section 2.1.

The proof of Theorem 1 is now complete.

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